9.5 VOLUME AND SURFACE AREA

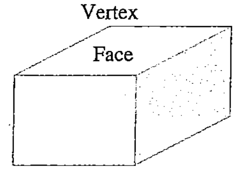
Space Figures • Volume and Surface Area of Space Figures

Polyhedral dice such as the ones shown here are often used in today's role-playing games.

The five regular polyhedra are also known as **Platonic solids,** named for the Greek philosopher Plato. He considered them as "building blocks" of nature and assigned fire to the tetrahedron, earth to the cube, air to the octahedron, and water to the icosahedron. Because the dodecahedron is different from the others due to its pentagonal faces, he assigned to it the cosmos (stars and planets). (*Source:* www.mathacademy.com) An animated view of the Platonic solids can be found at http://www.wikipedia.org/wiki/Platonic_solid

Space Figures

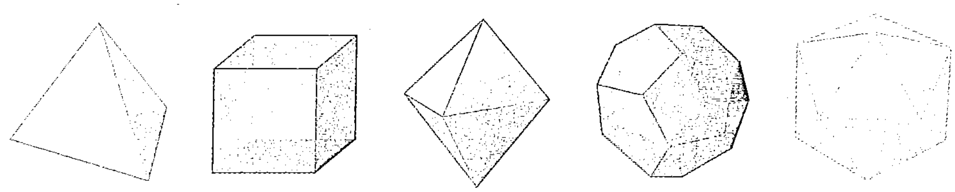
Thus far, this chapter has discussed only **plane figures**—figures that can be drawn completely in the plane of a sheet of paper. However, it takes the three dimensions of space to represent the solid world around us. For example, **Figure 53** shows a "box" (a **rectangular parallelepiped**). The *faces* of a box are rectangles. The faces meet at *edges;* the "corners" are called *vertices* (plural of vertex—the same word as for the "corner" of an angle).



Rectangular parallelepiped (box)

Figure 53

Boxes are one kind of space figure belonging to an important group called **polyhedra,** the faces of which are made only of polygons. Perhaps the most interesting polyhedra are the *regular polyhedra.* Recall that a *regular polygon* is a polygon with all sides equal and all angles equal. A regular polyhedron is a space figure, the faces of which are only one kind of regular polygon. It turns out that there are only five different regular polyhedra. They are shown in **Figure 54**. A **tetrahedron** is composed of four equilateral triangles, each three of which meet in a point. Use the figure to verify that there are four faces, four vertices, and six edges.



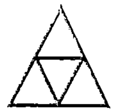
Tetrahedron Hexahedron (cube) Octahedron Dodecahedron Icosahedron

Figure 54

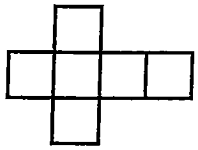
The four remaining regular polyhedra are the **hexahedron,** the **octahedron,** the **dodecahedron,** and the **icosahedron.** The hexahedron, or cube, is composed of six squares, each three of which meet at a point. The octahedron is composed of groups of four regular triangles (i.e., equilateral) meeting at a point. The dodecahedron is formed by groups of three regular pentagons, while the icosahedron is made up of groups of five regular triangles.

Two other types of polyhedra are familiar space figures: pyramids and prisms. **Pyramids** are made of triangular sides and a polygonal base. **Prisms** have two faces in parallel planes; these faces are congruent polygons. The remaining faces of a prism are all parallelograms. (See **Figures 55(a)** and **(b)** on the next page.) By this definition, a box is also a prism.

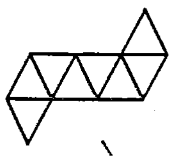
Figure 55(c) shows space figures made up in part of circles, including *right circular cones* and *right circular cylinders.* It also shows how a circle can generate a *torus,* a doughnut-shaped solid that has interesting topological properties. See **Section 9.7.**



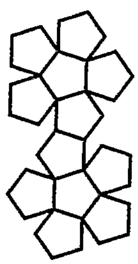
Tetrahedron



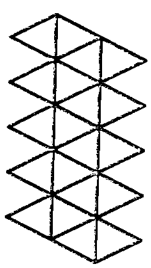
Hexahedron (cube)



Octahedron



Dodecahedron



Icosahedron

Patterns such as these may be used to construct three-dimensional models of the **regular polyhedra.** See

http://www.korthalsaltes.com

for some examples.

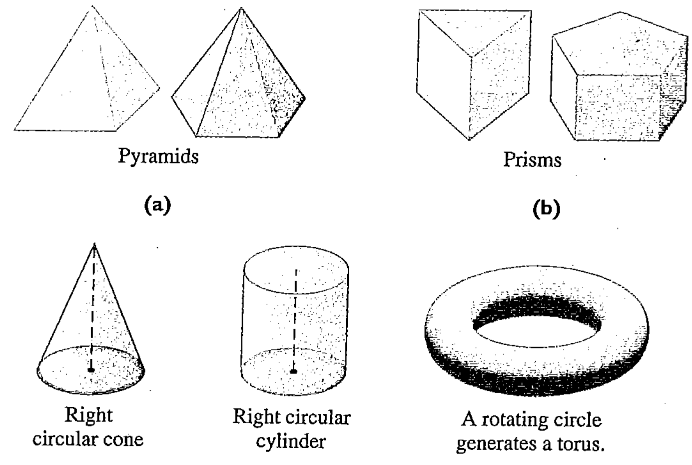


Figure 55

Volume and Surface Area of Space Figures

While area is a measure of surface covered by a plane figure, **volume** is a measur capacity of a space figure. Volume is measured in *cubic* units. For example, a cube edge measuring 1 cm has volume 1 cubic cm, which is also written as 1 cm^3, or 1 cc. **surface area** is the total area that would be covered if the space figure were "peel and the peel laid flat. Surface area is measured in *square* units.

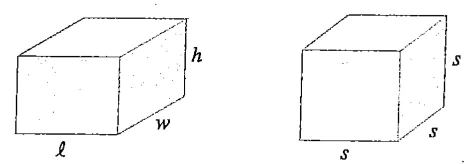
Volume and Surface Area of a Box

Suppose that a box has length ℓ, width w, and height h. Then the volume V anc the surface area S are given by the following formulas.

$$V = \ell wh \quad \text{and} \quad S = 2\ell w + 2\ell h + 2hw$$

If the box is a cube with edge of length s, the formulas are as follows.

$$V = s^3 \quad \text{and} \quad S = 6s^2$$



$V = \ell wh$

$S = 2\ell w + 2\ell h + 2hw$

$V = s^3$

$S = 6s^2$

EXAMPLE 1 Using the Formulas for a Box

Find the volume V and the surface area S of the box shown in **Figure 56**.

SOLUTION

$$\begin{aligned} V &= \ell wh && \text{Volume formula} \\ &= 14 \cdot 7 \cdot 5 && \text{Substitute.} \\ &= 490 && \text{Multiply.} \end{aligned}$$

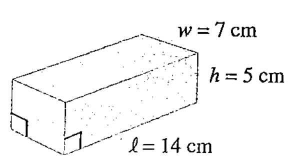


Figure 56

Volume is measured in cubic units, so the volume of the box is 490 cubic centime ters, or 490 cm^3.

$$S = 2\ell w + 2\ell h + 2hw \quad \text{Surface area formula}$$
$$= 2(14)(7) + 2(14)(5) + 2(5)(7) \quad \text{Substitute.}$$
$$= 196 + 140 + 70 \quad \text{Multiply.}$$
$$= 406 \quad \text{Add.}$$

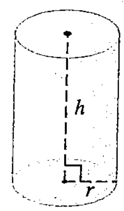
Surface areas of space figures are measured in square units, so the surface area of the box is 406 square centimeters, or 406 cm^2. ∎

John Conway of Princeton University offered a reward in the 1990s to anyone producing a **holyhedron,** a polyhedron with a finite number of faces and with a hole in every face. At the time, no one knew whether such an object could exist. When graduate student **Jade Vinson** arrived at Princeton, he immediately took up the challenge and in 2000 produced (at least the proof of the theoretical existence of) a holyhedron with 78,585,627 faces. The reward offered ($10,000 divided by the number of faces) earned Vinson $0.0001. Subsequently, **Don Hatch** produced one with 492 faces, good for a prize of $20.33. See this site for some interesting graphics: http://www.plunk.org/~hatch/Holyhedron

A typical tin can is an example of a **right circular cylinder.**

Volume and Surface Area of a Right Circular Cylinder

If a right circular cylinder has height h and radius of its base equal to r, then the volume V and the surface area S are given by the following formulas.



$V = \pi r^2 h$
$S = 2\pi rh + 2\pi r^2$

$$V = \pi r^2 h$$

and
$$S = 2\pi rh + 2\pi r^2$$

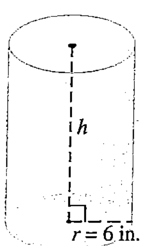
(In the formula for S, the areas of the top and bottom are included.)

EXAMPLE 2 Using the Formulas for a Right Circular Cylinder

In **Figure 57**, the right circular cylinder has surface area 288π square inches, and the radius of its base is 6 inches. Find each measure.

(a) the height of the cylinder

(b) the volume of the cylinder



Right circular cylinder

Figure 57

SOLUTION

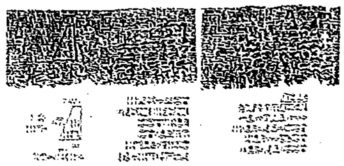
(a)
$$S = 2\pi rh + 2\pi r^2 \quad \text{Surface area formula}$$
$$288\pi = 2\pi(6)h + 2\pi(6)^2 \quad S = 288\pi,\ r = 6$$
$$288\pi = 12\pi h + 72\pi \quad \text{Multiply.}$$
$$216\pi = 12\pi h \quad \text{Subtract } 72\pi.$$
$$h = 18 \quad \text{Divide by } 12\pi.$$

The height is 18 inches.

(b)
$$V = \pi r^2 h \quad \text{Volume formula}$$
$$= \pi(6)^2(18) \quad r = 6,\ h = 18$$
$$= 648\pi \quad \text{Multiply.}$$

The exact volume is 648π in.^{3}, or approximately 2030 in.^{3}, using $\pi \approx 3.14$. ∎

The three-dimensional analogue of a circle is a **sphere.** It is defined by replacing the word "plane" with "space" in the definition of a circle (**Section 9.2**).



The **Moscow papyrus**, which dates back to about 1850 B.C., provides an example of inductive reasoning by the early Egyptian mathematicians. Problem 14 in the document reads:

You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take one-third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.

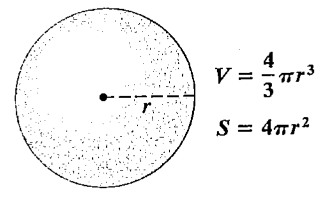
What does all this mean? A *frustum* of a pyramid is that part of the pyramid remaining after its top has been cut off by a plane parallel to the base of the pyramid. The formula for finding the volume of the frustum of a pyramid with square bases is

$$V = \frac{1}{3}h(b^2 + bB + B^2),$$

where b is the length of the upper base, B is the length of the lower base, and h is the height (or altitude). The writer of the problem is giving a method of determining the volume of the frustum of a pyramid with square bases on the top and bottom, with bottom base side of length 4, top base side of length 2, and height equal to 6.

Volume and Surface Area of a Sphere

If a sphere has radius r, then the volume V and the surface area S are given by the following formulas.



$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2$$

EXAMPLE 3 Using the Volume Formula for a Sphere

Suppose that a spherical tank having radius 3 meters can be filled with liquid for \$200. How much will it cost to fill a spherical tank of radius 6 meters with same fuel?

SOLUTION

Find the first volume, V_1.

$$\begin{aligned} V_1 &= \frac{4}{3}\pi r^3 && \text{Formula} \\ &= \frac{4}{3}\pi(3)^3 && r = 3 \\ &= \frac{4}{3}\pi(27) && \text{Cube.} \\ &= 36\pi && \text{Multiply.} \end{aligned}$$

Find the second volume, V_2.

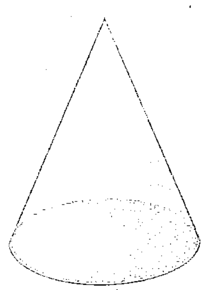
$$\begin{aligned} V_2 &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(6)^3 && r = 6 \\ &= \frac{4}{3}\pi(216) \\ &= 288\pi \end{aligned}$$

The volumes are 36π and 288π m^3. Notice that by doubling the radius of the sph from 3 meters to 6 meters, the volume has increased 8 times, because

$$V_2 = 288\pi = 8(36\pi) = 8V_1.$$

Therefore, the cost to fill the larger tank is eight times the cost to fill the smaller c $8(\$200) = \$1600$.

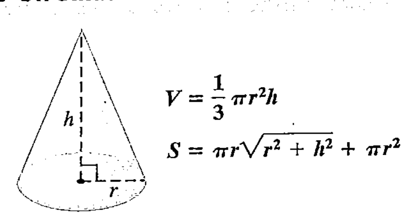
The space figure shown in **Figure 58** is a **right circular cone.**



Right circular cone
Figure 58

Volume and Surface Area of a Right Circular Cone

If a right circular cone has height h and the radius of its circular base is r, then the volume V and the surface area S are given by the following formulas.



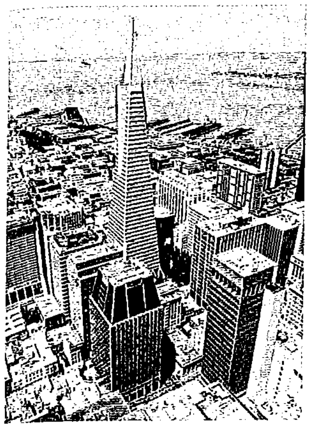
$$V = \frac{1}{3}\pi r^2 h$$

and $$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$$

(In the formula for S, the area of the bottom is included.)

Pyramid
Figure 59

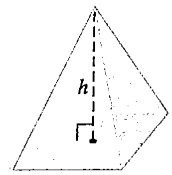
A **pyramid** is a space figure having a polygonal base and triangular sides. **Figure** shows a pyramid with a square base.



The **Transamerica Tower** in San Francisco is a pyramid with a square base. Each side of the base has a length of 52 meters, while the height of the building is 260 meters. The formula for the volume of a pyramid indicates that the volume of the building is about 234,000 cubic meters.

Volume of a Pyramid

If B represents the area of the base of a pyramid, and h represents the height (that is, the perpendicular distance from the top, or apex, to the base), then the volume V is given by the following formula.



$$V = \frac{1}{3}Bh$$

EXAMPLE 4 Comparing Volumes Using Ratios

What is the ratio of the volume of a right circular cone with radius of base r and height h to the volume of a pyramid having a square base, with each side of length r, and height h?

SOLUTION

First, use the formula for the volume of a cone.

$$V_1 = \text{Volume of the cone} = \frac{1}{3}\pi r^2 h$$

Because the pyramid has a square base, the area B of its base is r^2. Now use the formula for the volume of a pyramid.

$$V_2 = \text{Volume of the pyramid} = \frac{1}{3}Bh = \frac{1}{3}(r^2)h$$

Now find the ratio of the first volume to the second.

$$\frac{V_1}{V_2} = \frac{\frac{1}{3}\pi r^2 h}{\frac{1}{3}r^2 h} = \pi \quad \text{The ratio is } \pi.$$

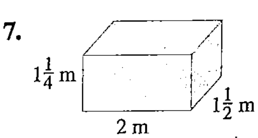
9.5 EXERCISES

Decide whether each statement is true *or* false.

1. A cube with volume 64 cubic inches has surface area 96 square inches. true
2. A tetrahedron has the same number of faces as vertices. true
3. A dodecahedron can be used as a model for a calendar for a given year, where each face of the dodecahedron contains a calendar for a single month, and there are no faces left over. true
4. Each face of an octahedron is an octagon. false
5. If you double the length of the edge of a cube, the new cube will have a volume that is twice the volume of the original cube. false
6. The numerical value of the volume of a sphere is $\frac{r}{3}$ times the numerical value of its surface area, where r is the measure of the radius. true

Find **(a)** *the volume and* **(b)** *the surface area of each space figure. When necessary, use* 3.14 *as an approximation for* π, *and round answers to the nearest hundredth.*

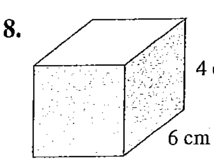
7.



(a box)

(a) $3\frac{3}{4}$ m^3 (b) $14\frac{3}{4}$ m^2

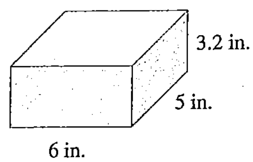
8.



(a box)

(a) 96 cm^3 (b) 128 cm^2

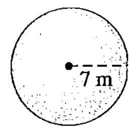
9.



(a box)

(a) 96 in.^{3} (b) 130.4 in.^{2}

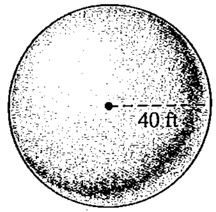
10.



(a sphere)

(a) 1436.03 m^3 (b) 615.44 m^2

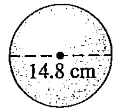
11.



(a sphere)

(a) 267,946.67 ft^3 (b) 20,096 ft^2

12.

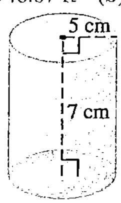


(a sphere)

(a) 1696.54 cm^3

(b) 687.79 cm^2

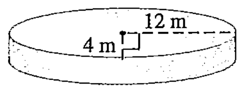
13.



(a right circular cylinder)

(a) 549.5 cm^3 (b) 376.8 cm^2

14.

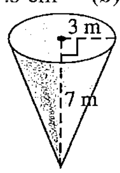


(a right circular cylinder)

(a) 1808.64 m^3

(b) 1205.76 m^2

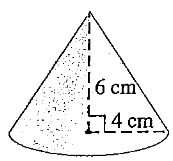
15.



(a right circular cone)

(a) 65.94 m^3 (b) 100.00 m^2

16.

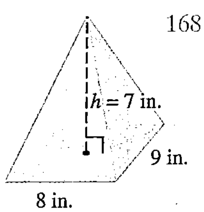


(a right circular cone)

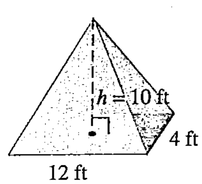
(a) 100.48 cm^3 (b) 140.81 cm^2

Find the volume of each pyramid. In each case, the base is a rectangle.

17. 168 in.^{3}



18. 160 ft^3



Volumes of Common Objects *Find each volume. Use 3.14 as an approximation for π when necessary.*

19. a coffee can, radius 6.3 cm and height 15.8 cm 1969.10 cm^3

20. a soup can, radius 3.2 cm and height 9.5 cm 305.46 cm^3

21. a pork-and-beans can, diameter 7.2 cm and height 10.5 cm 427.29 cm^3

22. a cardboard mailing tube, diameter 2 in. and height 40 in. 125.6 in.^{3}

23. a coffee mug, diameter 9 cm and height 8 cm 508.68 cm^3

24. a bottle of glue, diameter 3 cm and height 4.3 cm 30.38 cm^3

25. the Great Pyramid of Cheops, near Cairo—its base is a square 230 m on a side, while the height is 137 m 2,415,766.67 m^3

26. a hotel in the shape of a cylinder with a base radius of 46 m and a height of 220 m 1,461,732.8 m^3

27. a road construction marker, a cone with height 2 m and base radius $\frac{1}{2}$ m 0.52 m^3

28. the conical portion of a witch's hat for a Halloween costume, with height 12 in. and base radius 4 in. 200.96 in.^{3}

In the chart below, one of the values r (radius), d (diameter), V (volume), or S (surface area) is given for a particular sphere. Find the remaining three values. Leave π in your answers.

	r	d	V	S
29.	6 in.	12 in.	288π in.^{3}	144π in.^{2}
30.	9 in.	18 in.	972π in.^{3}	324π in.^{2}
31.	5 ft	10 ft	$\frac{500}{3}\pi$ ft^3	100π ft^2
32.	20 ft	40 ft	$\frac{32,000}{3}\pi$ ft^3	1600π ft^2
33.	2 cm	4 cm	$\frac{32}{3}\pi$ cm^3	16π cm^2
34.	4 cm	8 cm	$\frac{256}{3}\pi$ cm^3	64π cm^2
35.	1 m	2 m	$\frac{4}{3}\pi$ m^3	4π m^2
36.	6 m	12 m	288π m^3	144π m^2

Solve each problem.

37. ***Volume or Surface Area?*** In order to determine the amount of liquid a spherical tank will hold, would you need to use volume or surface area? volume

38. ***Volume or Surface Area?*** In order to determine the amount of leather it would take to manufacture a basketball, would you need to use volume or surface area? surface area

39. ***Side Length of a Cube*** One of the three famous construction problems of Greek mathematics required the construction of an edge of a cube with twice the volume of a given cube. If the length of each side of the given cube is x, what would be the length of each side of a cube with twice the original volume? $\sqrt[3]{2}\,x$

40. Work through the parts of this exercise in order, and use them to make a generalization concerning volumes of spheres. Leave answers in terms of π.

(a) Find the volume of a sphere having radius of 1 m. $\frac{4}{3}\pi$ m^3

(b) Suppose the radius is doubled to 2 m. What is the volume? $\frac{32}{3}\pi$ m^3

(c) When the radius was doubled, by how many times did the volume increase? (To find out, divide the answer for part (b) by the answer for part (a).) 8 times

(d) Suppose the radius of the sphere from part (a) is tripled to 3 m. What is the volume? 36π m^3

(e) When the radius was tripled, by how many times did the volume increase? 27 times

(f) In general, if the radius of a sphere is multiplied by n, the volume is multiplied by $\underline{\ n^3\ }$.

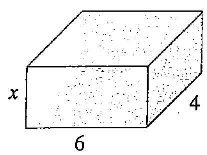
Cost to Fill a Spherical Tank *If a spherical tank* 2 m *in diameter can be filled with a liquid for* \$300, *find the cost to fill tanks of each diameter.*

41. 6 m \$8100 **42.** 8 m \$19,200 **43.** 10 m \$37,500

44. Use the logic of **Exercise 40** to answer the following: If the radius of a sphere is multiplied by n, then the surface area of the sphere is multiplied by $\underline{\ n^2\ }$.

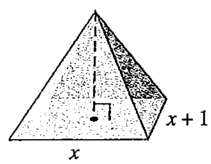
Each of the following figures has volume as indicated. Find the value of x.

45. $V = 60$ 2.5



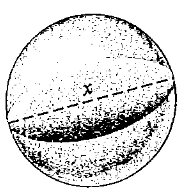
(a box)

46. $V = 450$ 9



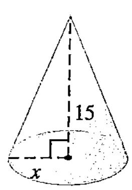
$h = 15$
Base is a rectangle.
(a pyramid)

47. $V = 36\pi$ 6



(a sphere)

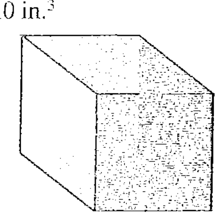
48. $V = 245\pi$ 7



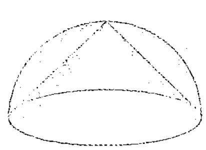
(a right circular cone)

Exercises 49–56 require some ingenuity, but all can be solved using the concepts presented so far in this chapter.

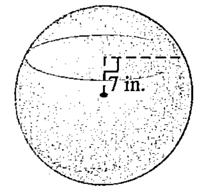
49. ***Volume of a Box*** The areas of the sides of a rectangular box are 30 in.^{2}, 35 in.^{2}, and 42 in.^{2}. What is the volume of the box? 210 in.^{3}



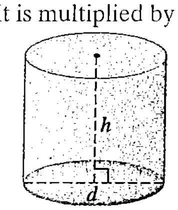
50. ***Ratios of Volumes*** In the figure, a right circular cone is inscribed in a hemisphere. What is the ratio of the volume of the cone to the volume of the hemisphere? 1 to 2



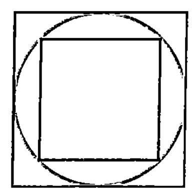
51. ***Volume of a Sphere*** A plane intersects a sphere to form a circle as shown in the figure. If the area of the circle formed by the intersection is 576 π in.^{2}, what is the volume of the sphere? $\frac{62{,}500}{3}\pi$ in.^{3}



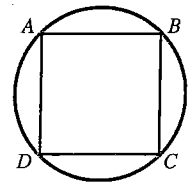
52. ***Change in Volume*** If the height of a right circular cylinder is halved and the diameter is tripled, how is the volume changed? It is multiplied by 4.5.



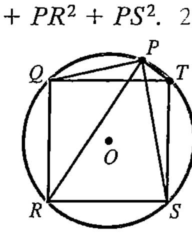
53. ***Ratio of Area*** What is the ratio of the area of the circumscribed square to the area of the inscribed square? 2 to 1



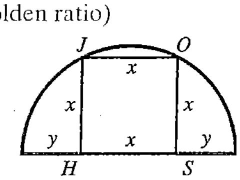
54. ***Perimeter of a Square*** Suppose the diameter of the circle shown is 8 in. What is the perimeter of the inscribed square $ABCD$? $16\sqrt{2}$ in.



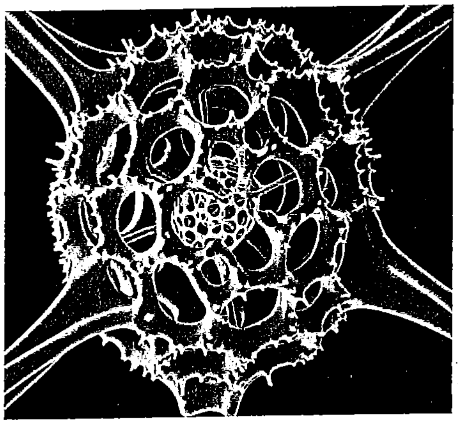
55. ***Value of a Sum*** In the circle shown with center O, the radius is 6. $QTSR$ is an inscribed square. Find the value of $PQ^2 + PT^2 + PR^2 + PS^2$. 288



56. ***Ratio of Side Lengths*** The square $JOSH$ is inscribed in a semicircle. What is the ratio of x to y?
$\frac{1+\sqrt{5}}{2}$ (the golden ratio)



Euler's Formula *Many crystals and some viruses are constructed in the shapes of regular polyhedra.*



Radiolara virus

Leonhard Euler investigated a remarkable relationship among the numbers of faces (F), vertices (V) and edges (E) for the five regular polyhedra. Complete the chart in Exercises 57–61, and then draw a conclusion in Exercise 62.

	Polyhedron	Faces (F)	Vertices (V)	Edges (E)	Value of $F + V - E$
57.	Tetrahedron	4	4	6	2
58.	Hexahedron (Cube)	6	8	12	2
59.	Octahedron	8	6	12	2
60.	Dodecahedron	12	20	30	2
61.	Icosahedron	20	12	30	2

62. Euler's formula is $F + V - E =$ __2__.

9.6 TRANSFORMATIONAL GEOMETRY

Reflections • Translations and Rotations • Size Transformations

In this chapter we have studied concepts of Euclidean geometry. Another branch of geometry, known as **transformational geometry,** investigates how one geometric figure can be transformed into another. In transformational geometry we are required to reflect, rotate, and change the size of figures using concepts that we now discuss.

Reflections

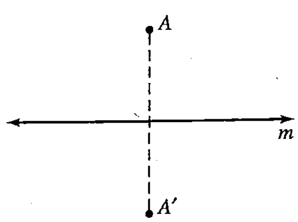
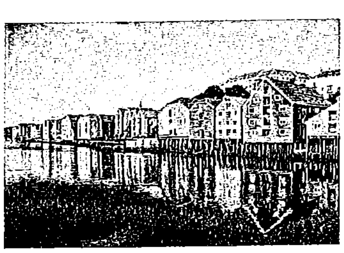


Figure 60

One way to transform one geometric figure into another is by reflection. In **Figure 60,** line m is perpendicular to the line segment AA' and bisects this line segment. We call point A' the **reflection image** of point A about line m. Line m is called the **line of reflection** for points A and A'. In the figure, we use a dashed line to connect points A and A' to show that these two points are images of each other under this transformation.

Point A' is the reflection image of point A only for line m. If a different line were used, A would have a different reflection image. Think of the reflection image of a point A about a line m as follows: Place a drop of ink at point A, and fold the paper along line m. The spot made by the ink on the other side of m is the reflection image of A. If A' is the image of A about line m, then A is the image of A' about the same line m.

To find the reflection image of a figure, find the reflection image of each point of the figure. The set of all reflection images of the points of the original figure is called the **reflection image** of the figure. **Figure 61** shows several figures (in black) and their reflection images (in color) about the lines shown.



An example of a **reflection.**

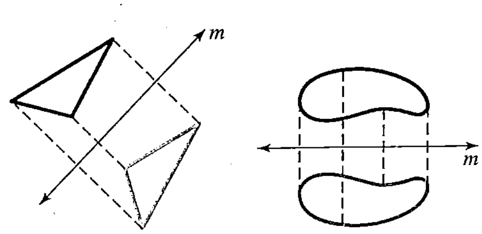


Figure 61