Syllabus

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Description of Topics: Our goal is to learn about exponents and units. We will do so primarily by working a lot of problems, as this is the best way to learn mathematics. Exponents and units are both topics highlighted in the common core. Exponents, or powers arise through repeated multiplication. Specifically, by multiplying a quantity by it self some number of times. Units are how we measure quantities. Both exponents and units are fundamental in the application of mathematics to science, technology, engineering, etc. For example, consider Newton's famous "inverse square law" for gravitational attraction which states that the magnitude of the gravitational attraction between two objects is inversely proportional to the distance between the two objects. In symbols

$$F = G\frac{mM}{r^2},$$

where F is the magnitude of the force, m and M are the masses of the two objects, r is the distance between the objects, and G is a constant of proportionality. Two observations:

1. The distance r is squared, *i.e.*, raised to the second power. Hence, we already see the need for exponents when applied mathematics to study nature.

2. When we measure physical quantities, take distance for example, we don't measure pure numbers, but numbers with some corresponding unit such as inches or meters. In order to communicate the significance of any measurements you make, you must state the unit, and know how to convert from one particular unit to another.

Mathematically, there is actually a lot of deep stuff going on with both exponents and units and we will try to explore some of this "deep stuff." This will give us a "view from the top" and increase our content knowledge of these topics that appear as part of the common core. Moreover, a fuller understanding of the things we teach will make us better equipped to teach those things.

About You

Date: Name: ______________________________

1. What do you enjoy most about mathematics?

2. What do you enjoy most about teaching mathematics?

3. What do you find is the greatest difficulty in teaching mathematics?

4. List a particular topic or area in mathematics that you would like to know more about.

5. What is the highest level of mathematics you have studied?

6. What can I do to ensure that you get the most out of the program?

1 Exponents

I

1. Evaluate
$$2^{-3}+2^{-5}.$$

I

2. Evaluate
$$\frac{2^{-2}}{3^{-3}}.$$

I

3. Evaluate
$$\left(\frac{2}{3}\right)^{-2}.$$

I

4. Simplify
$$121^{\frac{1}{2}}.$$

I

5. Simplify
$$(-8)^{\frac{2}{3}}(36)^{-\frac{3}{2}}.$$

I

6. Simplify and express answer without writing a fraction
$$\frac{a^2x^3y^4}{a^4x^3y^2}.$$

I

7. Simplify and express answer without writing a fraction
$$\frac{(x+y)^4}{(x+y)^5}.$$

I

8. Simplify and eliminate negative exponents.
$$b^4(3ab^3)(2a^2b^{-5})$$

I

9. Simplify and eliminate negative exponents.
$$\frac{6y^3z}{2yz^2}$$

I

10. Simplify and eliminate negative exponents.
$$\left(\frac{a^2}{b}\right)^5\left(\frac{a^3b^2}{c^3}\right)^3$$

I

11. Simplify and eliminate negative exponents.
$$(8y^3)^{-\frac{2}{3}}$$

I 12. Simplify and eliminate negative exponents.

$$\frac{(8s^3t^3)^{\frac{2}{3}}}{(s^4t^{-8})^{\frac{1}{4}}}$$

I 13. Simplify and eliminate negative exponents.

$$\left(\frac{x^{-\frac{2}{3}}}{y^{\frac{1}{2}}}\right)\left(\frac{x^{-2}}{y^{-3}}\right)^{\frac{1}{6}}$$

I 14. Simplify the expression

$$\frac{x^2y^3z}{xyz^2 + x^2y^{\frac{1}{2}}z^3}$$

I 15. Simplify the expression

$$\frac{\sqrt{x} - \frac{2}{\sqrt{x}}}{x^{\frac{3}{2}}}$$

I 16. Factor

$$4y^2(2y^2-1)^{-\frac{1}{3}} - 8y^4(2y^2-1)^{-\frac{4}{3}}$$

I 17. Factor

$$x^{\frac{2}{3}} - 2x^{-\frac{1}{3}} + x^{-\frac{4}{3}}$$

I 18. Simplify

$$\frac{x^{\frac{2}{3}} - 2x^{-\frac{1}{3}} + x^{-\frac{4}{3}}}{x^2-1}$$

I 19. Simplify

$$\frac{(y+4)^{\frac{2}{3}} - (y+4)^{-\frac{4}{3}}}{(y+4)^{\frac{2}{3}}}$$

I 20. Simplify

$$\frac{(y^5+2y^4)^{\frac{1}{2}} - (16y+32)^{\frac{1}{2}}}{y-2}$$

III 21. Compute the value without using a calculator

$$\frac{(1.0\times 10^{-1})(4.0\times 10^2)}{2.0\times 10^2}$$

III 22. Compute the value without using a calculator

$$\frac{(1.0\times 10^{-2})(4.0\times 10^4)}{2.0\times 10^{-2}}$$

III 23. Compute the value without using a calculator

$$2.0\times 10^2 + 1.0\times 10^3$$

III

24. Compute the value without using a calculator

$$2.0 \times 10^{2} + 1.0 \times 10^{-1}$$

III

25. Compute the value without using a calculator

$$\frac{(1.0 \times 10^{-1})(4.0 \times 10^{2})}{(2.0 \times 10^{2})^{2}}$$

III

26. Compute the value without using a calculator

$$\frac{(1.0 \times 10^{-1})(4.0 \times 10^{2})}{(2.0 \times 10^{2})^{\frac{1}{2}}}$$

III

27. Determine the order of magnitude for the following calculation

$$(1.2 \times 10^{-11})(5.7 \times 10^{13}).$$

Do not use a calculator.

III

28. Determine the order of magnitude for the following calculation

$$(3.3 \times 10^{-11})(4.7 \times 10^{13}).$$

Do not use a calculator.

III

29. Determine the order of magnitude for the following calculation

$$\frac{1.2 \times 10^{-11}}{5.7 \times 10^{13}}.$$

Do not use a calculator.

III

30. Determine the order of magnitude for the following calculation

$$\sqrt{\frac{(2.3 \times 10^{8})}{(1.7 \times 10^{6})(4.8 \times 10^{-2})}}$$

Do not use a calculator.

III

31. Determine the order of magnitude for the following calculation

$$2.3 \times 10^{12} + 1.8 \times 10^{-21}$$

Do not use a calculator.

III

32. Solve the following problem without using a calculator.

$$\frac{(2.3 \times 10^{7})(5.2 \times 10^{-5})}{4.3 \times 10^{2}}$$

a 1.2×10^{-1}

b 2.8

c 3.1×10

d 5.6×10^2

III

33. Solve the following problem without using a calculator.

$$((2.5 \times 10^{-7})(3.7 \times 10^{-6})) + 4.2 \times 10^2$$

a 1.3×10^{-11}

b 5.1×10^{-10}

c 4.2×10^2

d 1.3×10^{15}

III

34. Solve the following problem without using a calculator.

$$\sqrt{(1.1 \times 10^{-4}) + (8.9 \times 10^{-5})}$$

a 1.1×10^{-2}

b 1.4×10^{-2}

c 1.8×10^{-2}

d 2.0×10^{-2}

III

35. Solve the following problem without using a calculator.

$$\frac{1}{2}(3.4 \times 10^2)(2.9 \times 10^8)^2$$

a 1.5×10^{18}

b 3.1×10^{18}

c 1.4×10^{19}

d 3.1×10^{19}

III

36. Solve the following problem without using a calculator.

$$\frac{(1.6 \times 10^{-19}) \times 15}{36^2}$$

a 1.9×10^{-21}

b 2.3×10^{-17}

c 1.2×10^{-9}

d 3.2×10^{-9}

V

37. Recall that the imaginary number i satisfies $i^2 = -1$. Compute

(a) i^3,

(b) i^4.

Let n be a positive integer, conjecture a formula for i^n. Use mathematical induction to prove your conjecture.

V

38. Using the F.O.I.L. method, and the fact that $i^2 = -1$, compute
   (a) $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^3$
   (b) $\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3$

V

39. Use the identity
$$e^{ix} = \cos(x) + i\sin(x), \quad \text{where } i^2 = -1,$$
and the laws of exponents to derive the sum formulas
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B), \tag{1}$$
$$\sin(A+B) = \cos(A)\sin(B) + \sin(A)\cos(B). \tag{2}$$

V

IV also might be helpful

40. Recall that any complex number z can be written as $z = re^{i\theta}$. Then for any complex number z, by DeMoivre's formula, we have that, for any positive integer n
$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$
Use this to compute
$$\left(\sqrt{3} + i\right)^5.$$

41. Suppose that you start with a number $x > 0$, if you take x^n for increasing values of n starting at $n = 1$, will x^n increase or decrease?

42. Suppose that you start with a number $x > 0$, if you take $x^{\frac{1}{n}}$ for increasing values of n starting at $n = 1$, will x^n increase or decrease?

43. Explain why $x^0 = 1$ for any real number $x \neq 0$.

44. Can 0^0 be determined?

45. Using the definition
$$x^p = x \cdot x \cdots x \quad (p\text{-times}),$$
prove by induction that
$$x^n x^m = x^{n+m}.$$
Assume for simplicity that $x > 0$. What properties are numbers are essential for this property to be true?

VI

46. Suppose that A is a 2×2 matrix, *i.e.*,
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
How would you define A^n where n is a natural number?

VI 47. Suppose that A is a 2×2 matrix, *i.e.*,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If we define

$$A^p = A \cdot A \cdots A \quad (p\text{-times}),$$

where $A \cdot A$ is matrix multiplication is it still true that

$$A^n A^m = A^{n+m}?$$

48. Using the definition

$$x^p = x \cdot x \cdots x (p\text{-times}),$$

prove by induction that

$$(xy)^n = x^n y^n.$$

Assume for simplicity that $x, y > 0$. What properties of numbers are essential for this to be true?

VI 49. Suppose that A and B are 2×2 matrices. If we define

$$A^p = A \cdot A \cdots A \quad (p\text{-times}),$$

where $A \cdot A$ is matrix multiplication is it still true that

$$(AB)^n = A^n B^n?$$

VI 50. Consider a 2×2 matrix of the form

$$M = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

a matrix of this form represents a rotation of the plane through an angle θ and hence is sometimes referred to as a *rotation matrix*. Compute the following:

$$M^2, M^3.$$

It may help to recall that

$$\begin{aligned} \cos(2x) &= 2\cos^2(x) - 1, && (3) \\ \sin(2x) &= 2\sin(x)\cos(x), && (4) \\ \cos(3x) &= 4\cos^3(x) - 3\cos(x), && (5) \\ \sin(3x) &= 3\sin(x) - 4\sin^3(x). && (6) \end{aligned}$$

Conjecture a formula for M^n, where n is a natural number.

VI

51. Find a 2×2 matrix A such that

$$A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

In this case we say that A is the squareroot of

$$\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

VI

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$$A^2 = \begin{pmatrix} 4 & 1 \\ 0 & 9 \end{pmatrix}.$$

In this case, A is the square root of

$$\begin{pmatrix} 4 & 1 \\ 0 & 9 \end{pmatrix}.$$

53. Does every 2×2 matrix have a suare root?

54. Consider the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. Suppose we define "multiplication" to be the operation of addition. If $x \in \mathbb{Z}$, then what is the interpretation of "x raised to the power n", where n is a natural number? Do all of the usual laws of exponents still hold?

55. Suppose that $(G, \circ)$ is a group. That is, G is a set and $\circ$ is a binary operation on elements of G. For $g \in G$, how would you define g^n where n is a natural number? How would you define g^n, where n is an integer? Can you define g^p where p is a rational number?

56. What makes it possible to define x^p, where x is a real number and p is a rational number?

57. Consider the equation $x^2 = 1$, is there a unique real number solution to this equation?

58. Consider the equation $x^3 = 1$, is there a unique real number solution to this equation?

59. Consider the equation $z^3 = 1$, is there a unique complex number solution to this equation?

60. Recall that the composition $f \circ g$ of two functions f and g is defined by

$$(f \circ g)(x) = f(g(x)).$$

If we take "multiplication" to be function composition, then we can define raising to a power as

$$f^n(x) = (\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-times}})(x).$$

For example, if $f = \cos(x)$, then

$$f^3(x) = \cos(\cos(\cos(x))).$$

With respect to this notion of exponentiation, which of the usual laws of exponents continue to hold? Justify your anwers. Which laws fail, and why? **Note:** This notion of exponentiation is called iteration.

61. Consider the function $f(x) = x^2$, under the composition notion defined above. Compute the following:

$$f(0.5), f^2(0.5), f^3(0.5), \ldots, f^{10}(0.5).$$

Repeat this for the values $x = 1, 1.5$. How would you describe the behavior of the iteration of the function x^2? Conjecture how iterating this function for other values of x will behave.

62. Discuss interation for the function $f(x) = \cos(x)$.

2 Units

1. A restaurant chain has sold over 80 billion hamburger, if each hamburger is $\frac{1}{2}$ inch thick, how tall would a stack of these hamburgers be? The distance to the moon is approximately 240 thousand miles. How does the height of the 80 billion hamburgers compare with the distance to the moon.

2. If tuition and fees for a college course is approximately \$200 per credit hour, how much do you pay for a four credit course? If a typical four credit course meets four times a week, for fifty minutes at a time, for fifteen weeks, how much do you pay per minute of the class?

3. Suppose that the area of an object is 4ft^2, what is the area of this object if we want measurements in terms of inches?

4. Suppose that the volume of an object is 64in^3, what is the volume of this object if we want measurements in terms of feet?

5. Convert 15 feet to inches (there are 12 inches per foot).

6. Convert 6 gallons to quarts (there are 4 quarts in one gallon).

7. Convert 3 pints to gallons (there are 2 pints in a quart).

8. Suppose that p is pressure, and v is velocity. What are the dimensions of the equation

$$S = pv?$$

9. Explain why the following formula can not be the volume of a shape.

$$V = 8\pi h l r^2,$$

where h is a height, l is a length, and r is a radius.

II

10. Suppose that A is an area, l is a length, and a is an acceleration. What are the units for the expression
$$\sqrt{\frac{la}{A}}?$$

11. Suppose that $f(t)$ is a function of time t, and has units of length. What are the units for the derivative, $\frac{df}{dt}$, of f with respect to time?

12. If x has dimensions of length, and t has dimensions of time, what are the dimensions of $\frac{d^2x}{dt^2}$?

II

13. Given that m has units of mass, and a has units of acceleration, derive the dimensions for F if
$$F = ma.$$

II

14. Consider the Coulomb force law
$$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2},$$
where F is the force, q_1, q_2 are the charges of two particles, r is the distance between the particles, and ϵ_0 is the *permitivity of free space.* Use dimensional analysis to determine the dimensions of ϵ_0.

15. Suppose that $f(x)$ is a function of x, where x has units of length, and f has units of volume. What are the units for the derivative, $\frac{df}{dx}$, of f with respect to x?

16. Consider the differential equation
$$\frac{du}{d\tau} = \alpha u(1 - \frac{u}{K}). \tag{7}$$
Now define
$$x = \frac{u}{K}, \quad t = \alpha\tau.$$
Using the chain rule from calculus compute
$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = ?$$
Use this and equation (7) to derive a differential equation for x.

17. Consider the following system of differential equations:
$$\frac{dS}{d\tau} = -\alpha SI, \tag{8}$$
$$\frac{dI}{d\tau} = \alpha SI - \gamma I. \tag{9}$$
Define
$$x = \frac{S}{N}, \quad y = \frac{I}{N}, \quad t = \gamma\tau.$$
Use the chain rule from calculus to compute
$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = ?, \tag{10}$$
$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = ?, \tag{11}$$
$$\tag{12}$$
and derive differential equations for x and y.

II

18. Recall Newton's famous equation
$$F = ma,$$
where F is the force, m is mass, and a is the acceleration.

   (a) Suppose that we double the mass, how must we scale the acceleration in order to keep the force constant?

   (b) Suppose that we double the force, by how much must we scale the mass in order to keep form having to adjust the acceleration?

II

19. Consider the equation
$$F = \frac{kqQ}{r^2},$$
which gives the force induced by an electric charge. Here F is the force, k is a fixed constant, q, Q are the charges of two charged particles, and r is the distance between them.

   (a) If we increase the distance between the particles by a factor of 2, by how much must we adjust the charge Q to keep the force the same assuming all other variables are kept fixed?

   (b) If we increase the distance between the particles by a factor of 2, by how much must we adjust the force F assuming all other variables are kept fixed?

Appendix

A Useful Tables

Table 1: SI system of units for physical quantities.

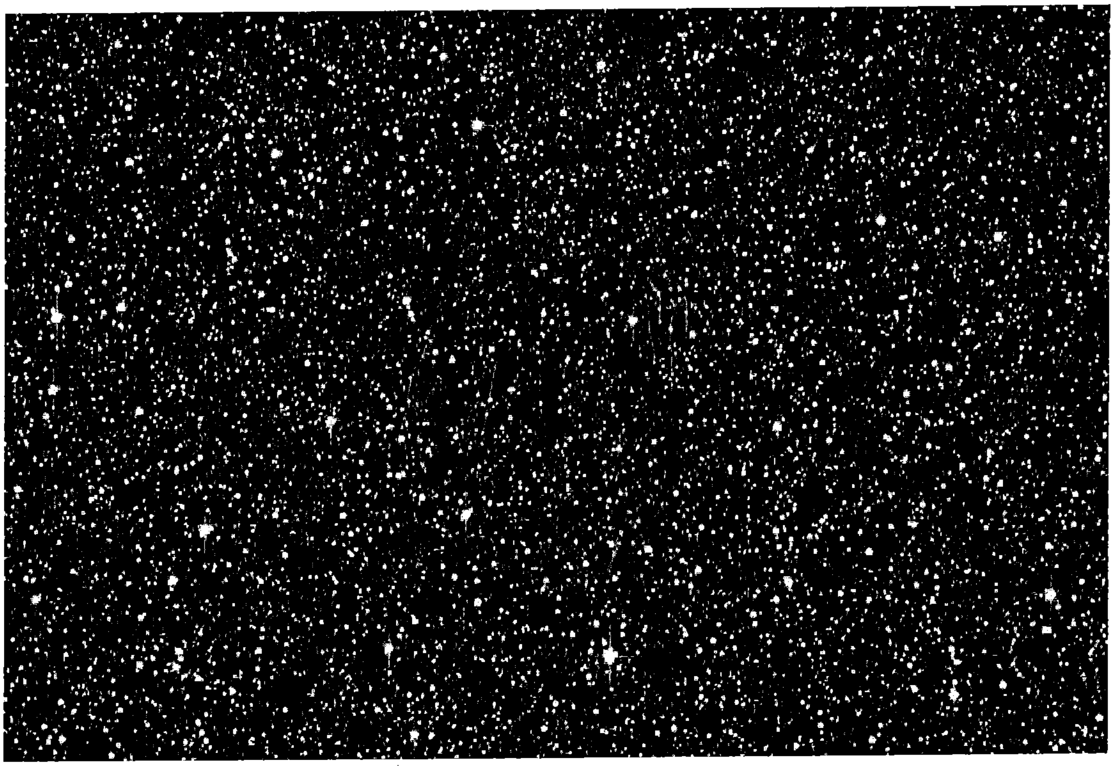
Quantity	Unit	Symbol
Length	meter	m
Time	second	s
Mass	kilogram	kg
Current	Ampere	A
Temperature	Kelvin	K
Amount	mole	mol

Table 2: Units and dimensions for various measurable quantities.

Quantity	Unit	Dimension Symbol
Length	meter m	L
Time	second s	T
Mass	kilogram kg	M
Electric Charge	coulomb C	I

Table 3: Prefixes for powers of ten.

10^{-1}	deci	d	10^{1}	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E
10^{-21}	zepto	z	10^{21}	zetta	Z
10^{-24}	yocto	y	10^{24}	yotta	Y



These delicate wisps of gas make up an object known as supernova remnant SNR 0519. The thin, blood-red shells are actually the remnants from when an unstable star exploded violently as a supernova around 600 years ago. SNR 0519 is located over 150,000 light-years from Earth in the southern constellation of Dorado, a constellation that also contains most of our neighboring galaxy, which is called the Large Magellanic Cloud. One light year equals about 10 trillion kilometers.

Problem 1 – The diameter of this supernova remnant shell is about 24 light years. The distance from our sun to the nearest star Alpha Centauri is 4.3 light years. If the sun were placed at the center of the supernova shell, about where would Alpha Centauri be at the same scale?

Problem 2 – The star that produced this shell exploded 600 years ago. If there are about 30 million seconds in one year, how fast was the shell traveling in kilometers/second expressed A) as a simplified fraction? B) As a decimal number?

http://www.nasa.gov/mission_pages/hubble/science/snr-0519.html

Hubble Sees the Remains of a Star Gone Supernova

May 3, 2013

Problem 1 – The diameter of this supernova remnant shell is about 24 light years. The distance from our sun to the nearest star Alpha Centauri is 4.3 light years. If the sun were placed at the center of the supernova shell, about where would Alpha Centauri be at the same scale?

Answer: **About 1/3 of the way from the center to the edge of the shell.**

Problem 2 – The star that produced this shell exploded 600 years ago. If there are about 30 million seconds in one year, how fast was the shell traveling in kilometers/second expressed A) as a simplified fraction? B) As a decimal number?

Answer: The center of the shell is 12 light years from the edge, so the distance traveled in 600 years is 12 x 10 trillion km, or 120 trillion kilometers.

Since 600 years equals 600x30 million seconds = 18 billion seconds,

the shell travels 120 trillion km / (18 billion seconds)
= 120,000 billion/18 billion
= 120,000/18

A) As a simplified fraction $\mathbf{\dfrac{20000}{3}}$ **kilometers per second**

B) As a decimal: 120000/18 = **6,666 kilometers per second.**

This speed is about 100 times faster than the Space Station orbiting Earth.

An image from an instrument aboard NASA's Landsat Data Continuity Mission or LDCM satellite may look like a typical black-and-white image of a dramatic landscape, but it tells a story of temperature. The dark waters of the Salton Sea are shown in the semi-circle on the left-hand edge of the image. Crops create a checkerboard pattern stretching south to the Mexican border.

The size of this image is 26 km wide and 17 km tall. Each green square represents a planted crop measuring 160 meters on a side and an area of about 6 acres.

Problem 1 - What percentage of the total area of this image is occupied by planted crops?

Problem 2 – What percentage of all the farmed areas actually have growing crops?

Problem 3 – The annual rain fall is about 3 inches per year (0.076 meters/yr). If one gallon of water has a volume of 0.0038 meters^3, how many gallons of water fall on the planted crop area each year?

New NASA Satellite Takes the Salton Sea's Temperature
April 22,2013
http://www.nasa.gov/mission_pages/landsat/news/salton-sea.html

Problem 1 - What percentage of the total area of this image is occupied by planted crops?

Answer: The total area of this image is 26 km x 17 km = 442 km^2.
Students should count the number of green squares to tally the number of planted areas. A typical number would be about 50, so the total planted area is 50 x 0.16 km x 0.16 km = 1.3 km^2. The percentage of the total area is then 100% x 1.3/442 = **0.3 %.**

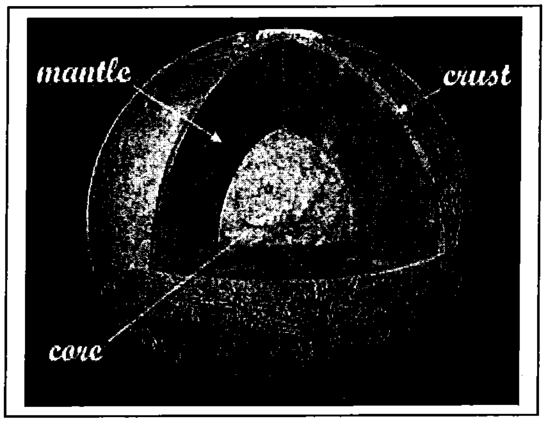
Problem 2 – What percentage of all the farmed areas actually have growing crops?

Answer: This is a bit more difficult because students have to count all of the square patches that they can see in the image, not just the green ones. A typical answer would be about 100 patches, so the total number of green + brown patches is about 150, and so the percentage of the planted areas is 100% x 50/150 = **33% or 1/3**.

Problem 3 – The annual rain fall is about 3 inches per year (0.076 meters/yr). If one gallon of water has a volume of 0.0038 $meters^3$, how many gallons of water fall on the planted crop area each year?

Answer:

From Problem 1, the total planted area is 1.3 km^2 or 1.3×10^6 $meters^2$. If the rain covers a depth of 0.076 meters each year, the rain volume is just 1.3×10^6 x 0.076 = 98800 cubic meters. This equals 98800 $meters^3$ x (1 gallon/0.0038 m^3) = **26 million gallons each year.**



Once astronomers have measured the diameter and mass of a planet, they can determine the average density of the planet by dividing its mass by its volume. This is a valuable 'first look' into the interior of a planet because if the average density is close to 1000 kg/m^3, then most of the planet consists of light materials and gas or even water and ice like Saturn and Uranus. If the value is large and near 4000 kg/m^3, then the planet may consist mostly of rocky materials like Mercury and Earth.

Problem 1 - The mass of Mars is known to be 6.39×10^{23} kilograms, and the outer radius of the planet is 3400 kilometers. What is the average density of Mars in kilograms/meter^3? What would you estimate as the composition of the martian interior if ice has a density of 917 kg/m^3, granite has a density of 2700 kg/m^3 and iron ore has a density of 7000 kg/m^3?

Problem 2 – The interior of mars can be represented by three main geologic regions: The core is a spherical region with a radius of about 1800 km; the mantle is a spherical shell with an outer radius of 3300 km, and the crust is a 100 km spherical shell located above the mantle. The crust of Mars has been sampled by several NASA landers including Viking, Spirit, Opportunity, Phoenix and Curiosity. The density of the surface rocks appears to be about 2000 kg/m^3. If models of the core of Mars suggest a density of 6400 kg/m^3, what is the average density of the rocks in the martian mantle zone to two significant figures?

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Answer: The volume of mars as a sphere is given by $V = 4/3 \pi R^3$ so
$V = 1.333 \times 3.141 \times (3400000)^3 = 1.65\times10^{20}$ m^3, then the density is just
$D = 6.39\times10^{23}$ kg/1.65×10^{20} m^3 = **3872 kg/m^3**. This is between the density of granite and iron, but closer to granite, so on average there is probably very little iron in the interior of Mars.

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Answer: We know:

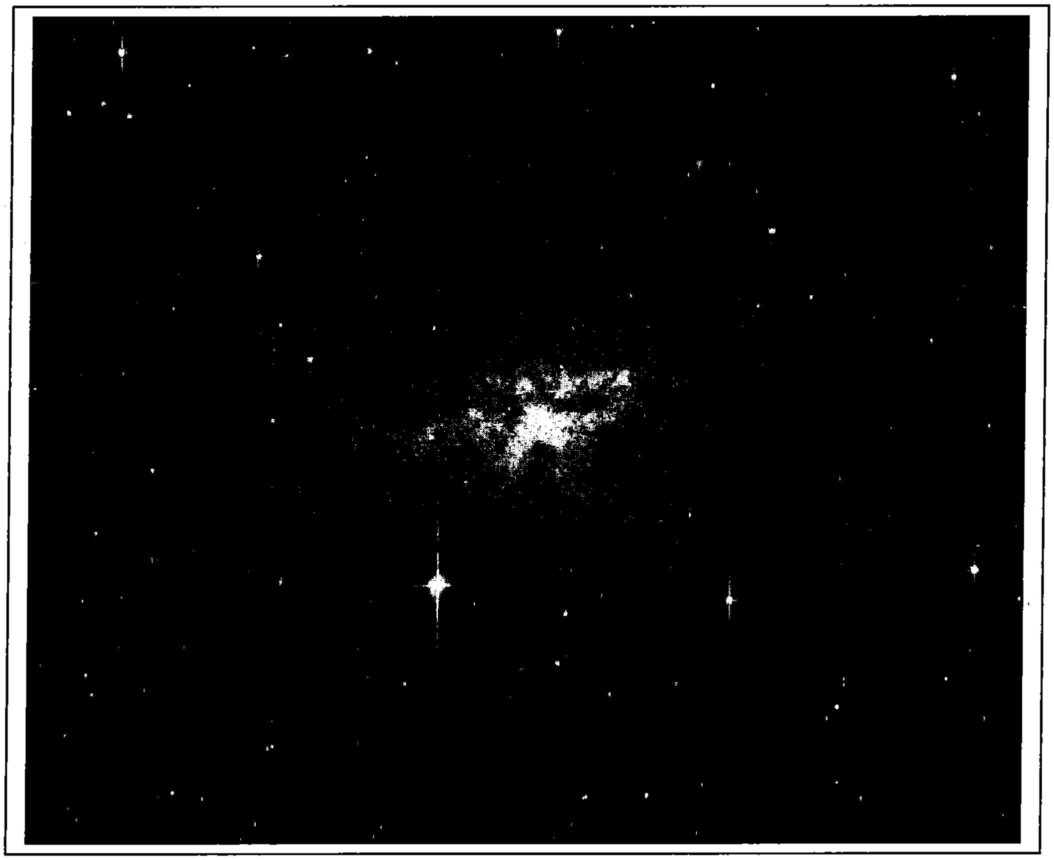
The total mass of mars is 6.39×10^{23} kg.
The radius of the core is 1800 km.
The inner and outer radius of the mantle shell as 1800 km and 3300 km.
The inner and outer radius of the crust shell as 3300 km and 3400 km.
The density of the core as 6400 kg/m^3
The density of the crust as 2000 kg/m^3.

So we subtract from the mass of Mars the mass of the core and the crust to get the mass of the mantle. From the mantle shell volume we can then determine it density:

Mcore = $6400 \times 4/3 \pi (1800000)^3 = 1.56\times10^{23}$ kg.
Mcrust = $2000 \times 4/3\pi (3400000^3 - 3300000^3) = 2.82\times10^{22}$ kg
Mmantle = 6.39×10^{23} kg – 1.56×10^{23} kg – 2.82×10^{22} kg = 4.55×10^{23} kg.
Volume(mantle) = $4/3\pi (3300000^3 - 1800000^3) = 1.26\times10^{20}$ m^3.

So Density = 4.55×10^{23} kg/1.26×10^{20}m^3 = **3600 kg/m^3**.

Giant Gas Cloud in System NGC 6240



Scientists have used Chandra to make a detailed study of an enormous cloud of hot gas enveloping two large, colliding galaxies. This unusually large reservoir of gas contains as much mass as 10 billion Suns, spans about 300,000 light years, and radiates at a temperature of more than 7 million degrees.

This giant gas cloud, which scientists call a "halo," is located in the system called NGC 6240. Astronomers have long known that NGC 6240 is the site of the merger of two large spiral galaxies similar in size to our own Milky Way. Each galaxy contains a supermassive black hole at its center. The black holes are spiraling toward one another, and may eventually merge to form a larger black hole.

Problem 1 - If 1 light year equals 9.5×10^{15} meters, and the cloud is in the shape of a sphere with a diameter of 300,000 light years, what is the volume of this cloud in cubic meters? ($\pi = 3.141$)

Problem 2 - The mass of the sun is 2.0×10^{30} kilograms. What is the density of this cloud in kilograms/m^3?

Problem 3 - If a single hydrogen atom has a mass of 1.7×10^{-27} kilograms, how many hydrogen atoms per cubic meter does the gas density represent?

Giant Gas Cloud in System NGC 6240
April 30, 2013
http://www.nasa.gov/mission_pages/chandra/multimedia/ngc6240.html

Problem 1 - If 1 light year equals 9.5×10^{15} meters, and the cloud is in the shape of a sphere with a diameter of 300,000 light years, what is the volume of this cloud in cubic meters? ($\pi = 3.141$)

Answer: $V = 4/3\ \pi\ R^3$ so
$V = 1.333\ (3.141)\ (150{,}000 \times 9.5\text{x}1015 \text{ meters})^3$
$= \mathbf{1.2 \times 10^{64}\ meters^3}$

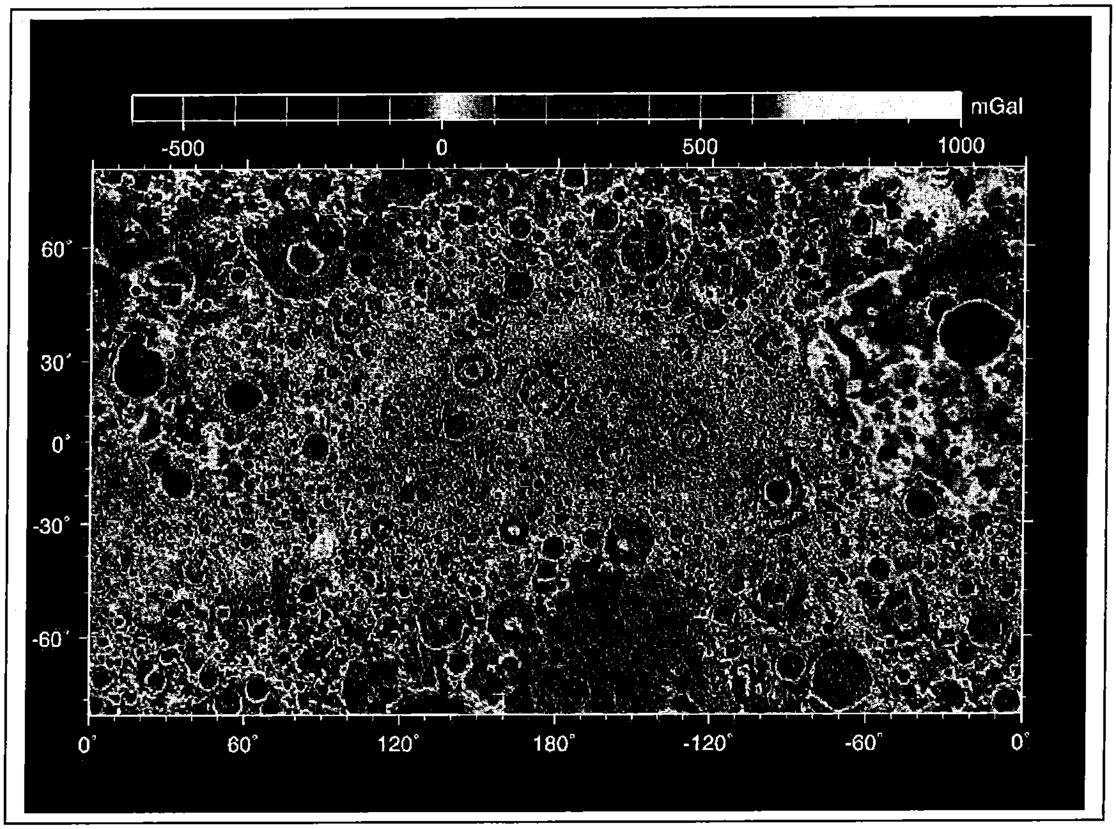
Problem 2 - The mass of the sun is 2.0×10^{30} kilograms. What is the density of this cloud in kilograms/m^3?

Answer: Mass of cloud = 10 billion suns = $10^{10} \times 2.0\text{x}10^{30}$ kg = $2.0\text{x}10^{40}$ kg.

Density = mass/volume
$= 2.0\text{x}10^{40}$ kg / $1.2\text{x}10^{64}$ m^3
$= \mathbf{1.7\text{x}10^{-24}\ kg/m^3}$

Problem 3 - If a single hydrogen atom has a mass of $1.7\text{x}10^{-27}$ kilograms, how many hydrogen atoms per cubic meter does the gas density represent?

Answer: Density = $1.7\text{x}10^{-24}$ kg/m^3 x (1 atom/$1.7\text{x}10^{-27}$ kg)
= **1000 atoms/meter^3**



During 2012, NASA's twin Grail satellites orbited the moon at altitudes of only 30 km. As they traveled, minute changes in their speeds tracked from Earth revealed changes in the gravitational field of the moon. These changes could be mapped, and revealed density changes in the lunar surface below them. In this way, scientists could look hundreds of kilometers beneath the lunar surface and explore how the surface was formed billions of years ago! On Earth, the acceleration of gravity is 9,807 cm/sec^2. The normal acceleration of gravity on the average lunar surface is 1620 cm/sec^2, but in the blue regions of the map this is as low as 1520 cm/sec^2, and in the red regions it is as high as 1920 cm/sec^2. A pendulum clock has a swinging period, T in seconds, given by the formula $T = 2\pi\sqrt{\frac{L}{g}}$ where L is the length of the pendulum in centimeters, and g is the acceleration of gravity in cm/sec^2.

Problem 1 - A lunar colony in a lunar 'blue' area has a Blue Clock with a pendulum length L = 100 cm. What is the swing period? (use π = 3.141)

Problem 2 - A lunar colony in a lunar 'red' area has an identical Red Clock. What is the swing period? (use π = 3.141)

Problem 3 - After how many swings on the Blue Clock will the clocks differ in time by 1 hour?

Problem 4 - If both clocks were synchronized to 1:00:00 am local time, what will the time on the Blue Clock and the Red Clock be when the two colony clocks are off by 1 hour relative to each other?

Problem 1 - A lunar colony in a lunar 'blue' area has a Blue Clock with a pendumum length L = 100 cm. What is the swing period?

Answer: $T = 2\ (3.141)\ (100/1520)^{1/2}$ = **1.61 seconds.**

Problem 2 - A lunar colony in a lunar 'red' area has an identical Red Clock. What is the swing period?

Answer; $T = 2\ (3.141)\ (100/1920)^{1/2}$ = **1.43 seconds.**

Problem 3 - After how many swings on the Blue Clock will the clocks differ in time by 1 hour?

Answer: Each swing on the slower Blue Clock pendulum is <u>behind</u> the faster Red Clock by 1.61 - 1.43 = 0.18 seconds. We want this difference to be 3600 seconds in 1 hour, which will take N = 3600/0.18 = **20,000 swings** on the Blue Clock.

Problem 4 - If both clocks were synchronized to 1:00:00 am local time, what will the time on the Blue Clock and the Red Clock be when the two colony clocks are off by 1 hour relative to each other?

Answer: On the Blue Clock, 20,000 swings have to pass, each taking 1.61 seconds for a total time of 32,200 seconds or 8 hours, 56 minutes, 40 seconds. So the time on the Blue Clock will read **09:56:40 am local time**.

On the Red Clock, because after 20,000 swings it is exactly 1 hour behind the Blue Clock, its time will read 08:56:40 am local time. Another 'long way' to see this is that we still need 20,000 swings to add up to a 1 hour time difference, but on the Red Clock each swing is only 1.43 seconds long and so this takes 28,600 seconds or 7 hours, 56 minutes, 40 seconds. The time on the Red Clock will be **08:56:40 am local time**.

This is why colonists will NOT be using pendulum clocks on the moon!!

Note: Devices that act like pendulum clocks were once used by prospectors on Earth to search for oil and other valuable materials below ground before the advent of more accurate magnetometer-based technology. Minute changes in the pendulum period indicate changes in the density of rock below ground and these can be used to identify high-gravity, density regions (like iron ore) or low-gravity regions (like caverns). Another way to measure minute gravity changes is by the shape of a satellite orbit, or by the subtle changes in speed between two satellites on the same orbit. Lunar scientists used this orbit method with the two Grail spacecraft only 200 kilometers apart.

I

Exponent Properties

1. **Product of like bases:** $a^m a^n = a^{m+n}$

   To multiply powers with the same base, add the exponents and keep the common base.

   Example: $x^5 x^3 = x^{5+3} = x^8$

2. **Quotient of like bases:** $\dfrac{a^m}{a^n} = a^{m-n}$

   To divide powers with the same base, subtract the exponents and keep the common base.

   Example: $\dfrac{x^5}{x^3} = x^{5-3} = x^2$

3. **Power to a power:** $(a^m)^n = a^{mn}$

   To raise a power to a power, keep the base and multiply the exponents.

   Example: $(x^5)^3 = x^{5*3} = x^{15}$

4. **Product to a power:** $(ab)^m = a^m b^m$

   To raise a product to a power, raise each factor to the power.

   Example: $(x^4 y^5)^3 = x^{12} y^{15}$

5. **Quotient to a power** $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

   To raise a quotient to a power, raise the numerator and the denominator to the power.

   Example: $\left(\dfrac{x^3}{y^2}\right)^4 = \dfrac{x^{12}}{y^8}$

6. **Zero Exponent:** $a^0 = 1$

   Any number raised to the zero power is equal to "1".

   Example: $(8x^4)^0 = 1$

7. **Negative exponent:** $a^{-n} = \dfrac{1}{a^n}$ or $\dfrac{1}{a^{-n}} = a^n$

   Negative exponents indicate reciprocation, with the exponent of the reciprocal becoming positive. You may want to think of it this way: unhappy (negative) exponents will become happy (positive) by having the base/exponent pair "switch floors"!

   Example: $8^{-2} = \dfrac{1}{8^2} = \dfrac{1}{64}$ *or* $\dfrac{4}{x^{-3}} = 4x^3$

Radicals - Rational Exponents

Objective: Convert between radical notation and exponential notation and simplify expressions with rational exponents using the properties of exponents.

When we simplify radicals with exponents, we divide the exponent by the index. Another way to write division is with a fraction bar. This idea is how we will define rational exponents.

Definition of Rational Exponents: $a^{\frac{n}{m}} = (\sqrt[m]{a})^n$

The denominator of a rational exponent becomes the index on our radical, likewise the index on the radical becomes the denominator of the exponent. We can use this property to change any radical expression into an exponential expression.

Example 1.

$(\sqrt[5]{x})^3 = x^{\frac{3}{5}}$	$(\sqrt[6]{3x})^5 = (3x)^{\frac{5}{6}}$	Index is denominator
$\dfrac{1}{(\sqrt[7]{a})^3} = a^{-\frac{3}{7}}$	$\dfrac{1}{(\sqrt[3]{xy})^2} = (xy)^{-\frac{2}{3}}$	Negative exponents from reciprocals

We can also change any rational exponent into a radical expression by using the denominator as the index.

Example 2.

$a^{\frac{5}{3}} = (\sqrt[3]{a})^5$	$(2mn)^{\frac{2}{7}} = (\sqrt[7]{2mn})^2$	Index is denominator
$x^{-\frac{4}{5}} = \dfrac{1}{(\sqrt[5]{x})^4}$	$(xy)^{-\frac{2}{9}} = \dfrac{1}{(\sqrt[9]{xy})^2}$	Negative exponent means reciprocals

World View Note: Nicole Oresme, a Mathematician born in Normandy was the first to use rational exponents. He used the notation $\frac{1}{3}\bullet 9^p$ to represent $9^{\frac{1}{3}}$. However his notation went largely unnoticed.

The ability to change between exponential expressions and radical expressions allows us to evaluate problems we had no means of evaluating before by changing to a radical.

Example 3.

$27^{-\frac{4}{3}}$ Change to radical, denominator is index, negative means reciprocal

$\dfrac{1}{(\sqrt[3]{27})^4}$ Evaluate radical

$\frac{1}{(3)^4}$ Evaluate exponent

$\frac{1}{81}$ Our solution

The largest advantage of being able to change a radical expression into an exponential expression is we are now allowed to use all our exponent properties to simplify. The following table reviews all of our exponent properties.

Properties of Exponents

$a^m a^n = a^{m+n}$	$(ab)^m = a^m b^m$	$a^{-m} = \frac{1}{a^m}$
$\frac{a^m}{a^n} = a^{m-n}$	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\frac{1}{a^{-m}} = a^m$
$(a^m)^n = a^{mn}$	$a^0 = 1$	$\left(\frac{a}{b}\right)^{-m} = \frac{b^m}{a^m}$

When adding and subtracting with fractions we need to be sure to have a common denominator. When multiplying we only need to multiply the numerators together and denominators together. The following examples show several different problems, using different properties to simplify the rational exponents.

Example 4.

$a^{\frac{2}{3}} b^{\frac{1}{2}} a^{\frac{1}{6}} b^{\frac{1}{5}}$ Need common denominator on $a's$ (6) and $b's$ (10)

$a^{\frac{4}{6}} b^{\frac{5}{10}} a^{\frac{1}{6}} b^{\frac{2}{10}}$ Add exponents on $a's$ and $b's$

$a^{\frac{5}{6}} b^{\frac{7}{10}}$ Our Solution

Example 5.

$\left(x^{\frac{1}{3}} y^{\frac{2}{5}}\right)^{\frac{3}{4}}$ Multiply $\frac{3}{4}$ by each exponent

$x^{\frac{1}{4}} y^{\frac{3}{10}}$ Our Solution

Example 6.

$\frac{x^2 y^{\frac{2}{3}} \cdot 2x^{\frac{1}{2}} y^{\frac{5}{6}}}{x^{\frac{7}{2}} y^0}$ In numerator, need common denominator to add exponents

$\frac{x^{\frac{4}{2}} y^{\frac{4}{6}} \cdot 2x^{\frac{1}{2}} y^{\frac{5}{6}}}{x^{\frac{7}{2}} y^0}$ Add exponents in numerator, in denominator, $y^0 = 1$

$$\frac{2x^{\frac{5}{2}}y^{\frac{9}{6}}}{x^{\frac{7}{2}}}$$ Subtract exponents on x, reduce exponent on y

$$2x^{-1}y^{\frac{3}{2}}$$ Negative exponent moves down to denominator

$$\frac{2y^{\frac{3}{2}}}{x}$$ Our Solution

Example 7.

$$\left(\frac{25x^{\frac{1}{3}}y^{\frac{2}{5}}}{9x^{\frac{4}{5}}y^{-\frac{3}{2}}}\right)^{-\frac{1}{2}}$$ Using order of operations, simplify inside parenthesis first
Need common denominators before we can subtract exponents

$$\left(\frac{25x^{\frac{5}{15}}y^{\frac{4}{10}}}{9x^{\frac{12}{15}}y^{-\frac{15}{10}}}\right)^{-\frac{1}{2}}$$ Subtract exponents, be careful of the negative:
$\frac{4}{10}-\left(-\frac{15}{10}\right)=\frac{4}{10}+\frac{15}{10}=\frac{19}{10}$

$$\left(\frac{25x^{-\frac{7}{15}}y^{\frac{19}{10}}}{9}\right)^{-\frac{1}{2}}$$ The negative exponent will flip the fraction

$$\left(\frac{9}{25x^{-\frac{7}{15}}y^{\frac{19}{10}}}\right)^{\frac{1}{2}}$$ The exponent $\frac{1}{2}$ goes on each factor

$$\frac{9^{\frac{1}{2}}}{25^{\frac{1}{2}}x^{-\frac{7}{30}}y^{\frac{19}{20}}}$$ Evaluate $9^{\frac{1}{2}}$ and $25^{\frac{1}{2}}$ and move negative exponent

$$\frac{3x^{\frac{7}{30}}}{5y^{\frac{19}{20}}}$$ Our Solution

It is important to remember that as we simplify with rational exponents we are using the exact same properties we used when simplifying integer exponents. The only difference is we need to follow our rules for fractions as well. It may be worth reviewing your notes on exponent properties to be sure your comfortable with using the properties.

II

Units and Dimensions

Units and dimensions tend to cause untold amounts of grief to many chemists throughout the course of their degree. My hope is that by having a dedicated tutorial on them we can avoid this for Hertford chemists. It is very important that you understand everything in this tutorial and that you (eventually) find the associated problems quite straightforward. Please make sure you ask lots of questions if there are things you don't understand (this goes for all tutorials, of course).

SECTION A – Reading and notes

Read the material taken from the book 'Quantities, units and symbols in physical chemistry' (often called the 'Green Book'), and also the overview below. Read them in any order you like, and take notes if it helps you.

Physical quantitites

A physical quantity is the product of a numerical value and a unit.

(Physical quantity) = (numerical value) x (unit)

e.g. (mass of an average person) = (70) x (kg) Obviously, this is usually just written 70 kg.
(speed of light) = (2.99792458×10^8) x (ms^{-1})

If you are one of the many people who, up until now, has always thought of units as something you have to tack onto the end of your calculations, appreciating the significance of the above is even more important. In science, we are generally dealing with physical quantities, not with pure numbers (we'll leave that to the mathematicians). This means that virtually every number you write down should have units with it. The numerical value of a physical quantity will vary depending on what units you choose to use (for example, an energy of 1 kJ could equally well be expressed as 1000 J, or 6.242×10^{21} eV, or 2.294×10^{20} Hartree), which means that just writing down a number without also stating its units is completely meaningless. *Units are not optional!*

The good news is that by thinking of units in this way, all the calculations and conversions and conventions that you previously may have found tortuous and completely incomprehensible should suddenly become much more straightforward. All calculations to do with units now essentially just become very basic algebra. For example, the tick marks along the axis of a graph are generally only labelled with numerical values. The axis label must therefore be consistent with this. Rearranging the above equation gives (numerical value) = (physical quantity)/(unit), so axes should always be labelled to be consistent with this e.g. speed / ms^{-1}, or mass / kg. The alternative, often seen in publications from the US and written e.g. speed (ms) or mass (kg) is technically incorrect and unfortunately shows that the authors do not understand physical quantities.

Before we move onto calculations, we need a short recap of the SI (Système Internationale) system of units.

SI units

The SI system identifies base units. These are defined very precisely (see the Green Book material for details) and are independent of one another.

Quantity	Unit	Symbol
Mass	kilogram	kg
Length	metre	m
Time	second	s
Current	Ampere	A
Temperature	Kelvin	K
Amount	mole	mol

All other SI units can be expressed in terms of these base units. You can work out the definitions very easily if you know a definition of the quantity you're interested in.

For example, the SI unit of energy is the Joule (J). If we want to know how a Joule is defined in terms of the base units, we could use the definition of the kinetic energy of a moving object:

$E = \frac{1}{2} mv^2$, where m and v are the mass and velocity of the object.

You will be used to substituting numerical values into this type of equation, but really what you are doing is substituting in *physical quantities*. The only reason you don't usually substitute in the units with your numerical value is that the units part of the calculation is the same every time, so you already know the result (though you may not have realised this before!). Consider the kinetic energy of a 10 kg object travelling at 2 ms^{-1}.

$$E = \tfrac{1}{2}\,(10\ \mathrm{kg})(2\ \mathrm{ms^{-1}})^2 = 40\ \mathrm{kg\ m^2\ s^{-2}} = 40\ \mathrm{J}$$

We see that the units of J are equivalent to kg m^2 s^{-2}. If we're *just* interested in relationships between units, then we can just substitute the units into an equation (just as if we're just interested in the numerical result then we only substitute the numerical values into an equation). If we're just doing a units calculation then we can ignore constant factors (e.g. the factor of ½ in the equation for the kinetic energy).

As another example, consider the potential energy of an object in the gravitational field of the earth at a height h above the earth's surface.

$E = mgh$, where g is the acceleration due to gravity, 9.8 ms^{-2}.

A units calculation would therefore give:

$$\mathrm{J} = (\mathrm{kg})\,(\mathrm{ms^{-2}})\,(\mathrm{m}) = \mathrm{kg\ m^2\ s^{-2}}.$$

Reassuringly, this is the same result as before. Hopefully this convinces you that you can choose any equation you like to work out how to express an SI unit in terms of the base units.

Often, you will see SI units with prefixes, which denote powers of ten. You need to know these prefixes (at least up to powers of plus or minus 15).

10^{-1}	deci	d	10^{1}	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E
10^{-21}	zepto	z	10^{21}	zetta	Z
10^{-24}	yocto	y	10^{24}	yotta	Y

Calculations with physical quantities

There are a few very simple rules regarding calculations with physical quantities.

1. You can only add or subtract quantities with the same units e.g. 10 kg + 5 kg = 15 kg, while 10 kg + 400 m is completely nonsensical. Note: check that you have all energies in the same units before carrying out this type of calculation e.g. all in J or all in kJ, not a mixture of the two.

2. When you multiply or divide, the units multiply and divide with the quantities, as shown in the previous section.

3. The arguments of logs, exponentials, and other functions that may be expanded as power series may only be dimensionless numbers.

   e.g. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

   If x was not dimensionless, every term in the expansion would have different units!

   This can be very useful in helping us work out the units of physical quantities. For example, a first order radioactive decay can be described by the equation $n = n_0 e^{-kt}$, where n is the amount of substance, n_0 is the amount of substance at time zero, k is the rate constant for the decay, and t is time. If we didn't know the units for the rate constant, we could use the fact that the product kt must be dimensionless to work them out. Since we know that time has units of seconds, k must have units of s^{-1}.

Unit conversions

This is an area in which many students frequently get themselves in a complete tangle or despair completely. However, once you have the definition of a physical quantity clear in your head it is really very simple to convert between units.

As an example, consider the volume V of a cube with sides of length L.

$$V = L^3$$

In SI units, L would be given in m, and V would therefore be in m^3. Assume we have sides of length 2 m. This would give a volume of 8 m^3. However, what if we wanted to know the volume in cm^3? Simple: 1 m = 100 cm, so:

$$V = (2\ m)^3 = (2 \times 100\ cm)^3 = 8 \times 10^6\ cm^3$$

Consider a second example. Suppose we want to convert 324 kJ mol^{-1} into J $molecule^{-1}$. We know that 1 kJ = 1000 J, and that 1 mol = 6.022 x 10^{23} molecules. Therefore:

$$324 \text{ kJ mol}^{-1} = 324 \times (1000 \text{ J}) \times (6.022 \times 10^{23} \text{ molecules})^{-1} = 5.38 \times 10^{-19} \text{ J molecule}^{-1}$$

Dimensional analysis

Sometimes we can work out the form of an equation simply by knowing the units of the quantities involved. There is often only one combination of the quantities that is consistent with their units. As a very simple example, suppose somebody tells you that the speed of an object has units of ms^{-1}, and they know that you can work out the speed of an object from the distance it has travelled and the time it took to travel that distance. However, they can't remember the required equation. You can work it out by looking at the units:

Speed v has units of ms^{-1}
Distance d has units of m
Time t has units of s

The obvious combination of quantities with units of m and s to give a quantity with units of ms^{-1} is

$$v = d / t$$

We could have done this calculation in a more formal way by equating powers of units i.e.

$$(\text{speed}) = (\text{distance})^a (\text{time})^b$$

So in terms of units $(ms^{-1}) = (m)^a (s)^b$

We immediately see that $a = 1$ and $b = -1$, so speed = $(\text{distance})^1(\text{time})^{-1}$, or $v = d / t$ as before.

We can also go back to our kinetic energy example. Suppose you know that kinetic energy is measured in J (and that the equivalent in SI base units is kg m^2 s^{-2}, and you also know that the kinetic energy depends on the mass of the object and on its velocity, but you can't remember the relationship.

$$(\text{Energy}) = (\text{mass})^a (\text{velocity})^b$$

So in terms of units $(kg\ m^2\ s^{-2}) = (kg)^a (ms^{-1})^b$

It is very straightforward to see that $a = 1$ and $b = 2$. If it had been less straightforward we could have matched terms on the left and right hand sides of the equation

$$kg = kg^a$$
$$m^2 = m^b$$
$$s^{-2} = (s^{-1})^b$$

which again gives $a = 1$, $b = 2$. Our dimensional analysis therefore tells us that

$$E \propto mv^2$$

Dimensional analysis unfortunately can only give us the proportionalities between physical quantities. In this case it cannot give us the required factor of ½.

SECTION B – Problems

1. Identify the SI units for the following quantities, and use the accompanying expressions to express them in terms of SI base units.

   (a) Force $F = ma$, where F = force, m = mass, a = acceleration

   (b) Pressure $p = F/A$ where p = pressure, F = force, A = area

2. How many dm^3 are there in one m^3?

3. When a substance diffuses, the *flux* is defined as the rate at which the amount of substance diffuses per unit area. According to Fick's law of diffusion, the flux is equal to minus the diffusion coefficient, D, times the concentration gradient, dc/dx.

   (a) What are the correct SI units for the flux and the concentration gradient?

   (b) Hence deduce the SI units for the diffusion coefficient.

4. The universal gas constant, R, can be calculated from measurements of pressure, volume, temperature and amount of substance under ideal conditions from $R = pV/nT$.

   (a) Find the SI units for R.

   (b) 1 mol of gas occupies 24.8 m^3 at 298 K and 1.00 mbar. Calculate R.

   (c) What is the concentration of the gas? (mol dm^{-3} and molecules cm^{-3}).

5. Consider the following statement:

   "It is not permitted to take the log of a unit, so in the equation $\Delta G^{\ominus} = -RT \ln K$, the equilibrium constant has no units. The only equilibrium constants with no units are for equilibria with equal numbers of particles on each side of the reaction equation, and so the equation above is only meaningful for reactions of this type."

   Which of the following is the best statement of the flaw in this argument?

   A Units are always ignored when logarithms are taken.
   B The units of K depend on the relative numbers of reactants and products in the chemical equation.
   C In calculating K, it is necessary to use activities instead of concentrations, and activities are dimensionless.
   D There is no flaw in this argument.

6. A molecule of carbon dioxide occupies a volume of 3.2×10^{-26} m^3. In the British system the smallest unit of volume is the minim, which is equivalent to 0.05919385 cm^3. What is the volume of the molecule in minims?

7. The speed limit on a road in Rutland is 135000 furlongs per fortnight. Given that a furlong is 1/8 mile and a fortnight is 14 days, calculate the speed limit in miles per hour.

8. The slug is an American unit of mass equivalent to 14.5939 kg, and 1 foot = 30.48 cm (exactly). The density of a soil sample is 3.01 g cm^{-3}. Convert this density to slugs per cubic foot.

9. A solution of sodium chloride has concentration 0.15 mol dm^{-3}. Convert this concentration into molecules nm^{-3}.

10. (a) Express the SI units for density and pressure in terms of SI base units.

    (b) Gas escapes through a small hole in the side of a vessel. The rate of loss of mass depends on the pressure of the gas, its density and the area of the hole.

        (i) Use dimensional analysis to determine this dependence.

        (ii) If the gas is ideal, how will the rate of loss depend on the molecular weight at a given pressure and temperature?

11. The speed of sound in a gas can be expressed in terms of its pressure and its density.

    (a) Use dimensional analysis to determine this dependence.

    (b) If the gas is ideal, how will the speed of sound depend on the molecular weight at a given temperature?

    (c) The speed of sound in air at room temperature is 330 ms^{-1}. Calculate the speed of sound in gaseous helium at the same temperature.

12. When an oil droplet is released, it falls under the influence of gravity until it reaches its terminal velocity, at which the gravitational force exactly balances the frictional force exerted by the air through which it passes. The terminal velocity depends on the weight mg of the drop (g is the acceleration due to gravity and m is the mass of the droplet), the viscosity η of the medium, and the radius a of the droplet.

    (a) Use dimensional analysis to work out how the terminal velocity should depend on all of these factors.
    [The SI units of viscosity are kg m^{-1} s^{-1}.]

    (b) What will be the effect of the following changes on the terminal velocity?

        (i) Using a gas with twice the viscosity of air.

        (ii) Using an oil drop with double the radius.

13. The rotational energy of a diatomic molecule is a function of its bond length, r, its reduced mass, μ, and Planck's constant, h. Use dimensional analysis to find out how the energy depends on these quantities.

14. The wind chill factor is the reduction in temperature due to the wind speed. It arises from the conversion of random motion (temperature) into organised motion (wind). The wind chill factor ΔT depends on the wind speed, v, the molecular mass of the gas, m, and Boltzmann's constant k, which has the value 1.38×10^{-23} J K^{-1}. Find the dimensions of each of these quantities and use dimensional analysis to discover how ΔT depends on them.

15. The molecular collision frequency per unit concentration in a gas, Z, has units m^3 s^{-1} and depends on the Boltzmann constant, k_B, the temperature, T, the molecular mass, m, and the molecular diameter, d. Use dimensional analysis to determine how Z depends on these quantities.

Scientific Notation Notes

<table>
<tr><td>What is Scientific Notation?</td><td colspan="2">A way to write very large or very small numbers more easily (takes up less space and less likely to make mistakes). Often used by scientists.

3.45×10^9 instead of 3,450,000,000</td></tr>
<tr><td>Definition</td><td colspan="2">A number is in scientific notation when it has a number between 1 and 10 multiplied by 10 to a power.
Which of the following are in Scientific Notation? Explain.
4.69×10^{-8} 56.7×10^5
0.35×10^3 7.224×10^{-6}
*The circled items are in Scientific Notation because their numbers are between 1 and 10 (not bigger like 56.7 or smaller like 0.35).</td></tr>
<tr><td>How to put numbers into Scientific Notation.</td><td>Examples:
37,498,000,000

3.7498000000.

Correct answer is
3.7498×10^{10}

0.0000000492

0.00000004.92

Correct answer is
4.92×10^{-8}</td><td>1) Move the decimal point until you have a number between 1 and 10.
2) Count the number of digits you moved the decimal point over. This number becomes the power of 10.
3) If you moved the decimal point to the right, it is a negative power (you started with a really small decimal). If you moved the decimal point to the left, it is a positive power (you started with a really large number)
4) Rewrite the number with the decimal point where you moved it to and multiplied by 10^n.</td></tr>
<tr><td>How to change numbers from</td><td>Examples:
1.45×10^4</td><td>1. Move the decimal point the number of digits indicated by the</td></tr>
</table>

Scientific Notation to Standard Form.	1.4500. Correct answer is 14,500 2.07×10^{-5} 0.00002.07 Correct answer is 0.0000207	power on the 10. You may need to add some zeros. 2. If the power is positive, move the decimal point to the right. (This will give you a large number.) If the poser is negative, move the decimal point to the left. (This will give you a small decimal number.) 3. Double check that you counted the number of digit spaces to move the decimal point correctly—this is the most common mistake.

Name ______________________ Date ____________ Class ____________

Math Skills for Science

MATH SKILLS

Multiplying and Dividing in Scientific Notation

Part 1: Multiplying in Scientific Notation

PROCEDURE: To multiply numbers in scientific notation, multiply the decimal numbers. Then *add* the exponents of the powers of 10. Place the new power of 10 with the decimal in scientific notation form. If your decimal number is greater than 10, count the number of times the decimal moves to the left, and add this number to the exponent.

SAMPLE PROBLEM: Multiply (2.6×10^7) by (6.3×10^4).

Step 1: Multiply the decimal numbers.

$$2.6 \times 6.3 = 16.38$$

Step 2: Add the exponents.

$$7 + 4 = 11$$

Step 3: Put the new decimal number with the new exponent in scientific notation form.

$$16.38 \times 10^{11}$$

Step 4: Because the new decimal number is greater than 10, count the number of places the decimal moves to put the number between 1 and 10. Add this number to the exponent. In this case, the decimal point moves one place, so add 1 to the exponent.

$$16.38 \times 10^{11} \rightarrow 1.638 \times 10^{12}$$

Try It Yourself!

1. Follow the steps in the Sample Problem carefully to complete the following equations.

Multiplying with Scientific Notations

Problem	New decimal	New exponent	Answer
Sample problem: $(4.4 \times 10^6) \times (3.9 \times 10^4)$	$4.4 \times 3.9 = 17.16$	$6 + 4 = 10$	1.716×10^{11}
a. $(2.8 \times 10^8) \times (1.9 \times 10^4)$			
b. $(1.3 \times 10^9) \times (4.7 \times 10^{-5})$			
c. $(3.7 \times 10^{15}) \times (5.2 \times 10^7)$			
d. $(4.9 \times 10^{24}) \times (1.6 \times 10^5)$			

2. The mass of one hydrogen atom is 1.67×10^{-27} kg. A cylinder contains 3.01×10^{23} hydrogen atoms. What is the mass of the hydrogen?

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MATH SKILLS

Name ______________________________ Date ____________ Class ____________

Part 2: Dividing in Scientific Notation

PROCEDURE: To divide numbers in scientific notation, first divide the decimal numbers. Then *subtract* the exponents of your power of 10. Place the new power of 10 with the decimal in scientific notation form. If the resulting decimal number is less than 1, move the decimal point to the right and decrease the exponent by the number of places that the decimal point moved.

SAMPLE PROBLEM: Divide (1.23×10^{11}) by (2.4×10^{4}).

Step 1: Divide the decimal numbers. $1.23 \div 2.4 = 0.5125$	**Step 2:** Subtract the exponents of the powers of 10. $11 - 4 = 7$
Step 3: Place the new power of 10 with the new decimal in scientific notation form. 0.5125×10^{7}	**Step 4:** Because the decimal number is not between 1 and 10, move the decimal point one place to the right and decrease the exponent by 1. $0.5125 \times 10^{7} \rightarrow 5.125 \times 10^{6}$

$(1.23 \times 10^{11}) \div (2.4 \times 10^{4}) = \mathbf{5.125 \times 10^{6}}$

3. Complete the following chart:

Dividing with Scientific Notation

Problem	New decimal	New exponent	Answer
Sample problem: $(5.76 \times 10^{9}) \div (3.2 \div 10^{3})$	$5.76 \div 3.2 = 1.8$	$9 - 3 = 6$	1.8×10^{6}
a. $(3.72 \times 10^{8}) \div (1.2 \times 10^{5})$			
b. $(6.4 \times 10^{-4}) \div (4 \times 10^{6})$			
c. $(3.6 \times 10^{4}) \div (6 \times 10^{5})$			
d. $(1.44 \times 10^{24}) \div (1.2 \times 10^{17})$			

4. The average distance from Earth to the sun is 1.5×10^{11} m. The speed of light is 3×10^{8} m/s. Approximately how long does it take for light to travel from the sun to Earth?

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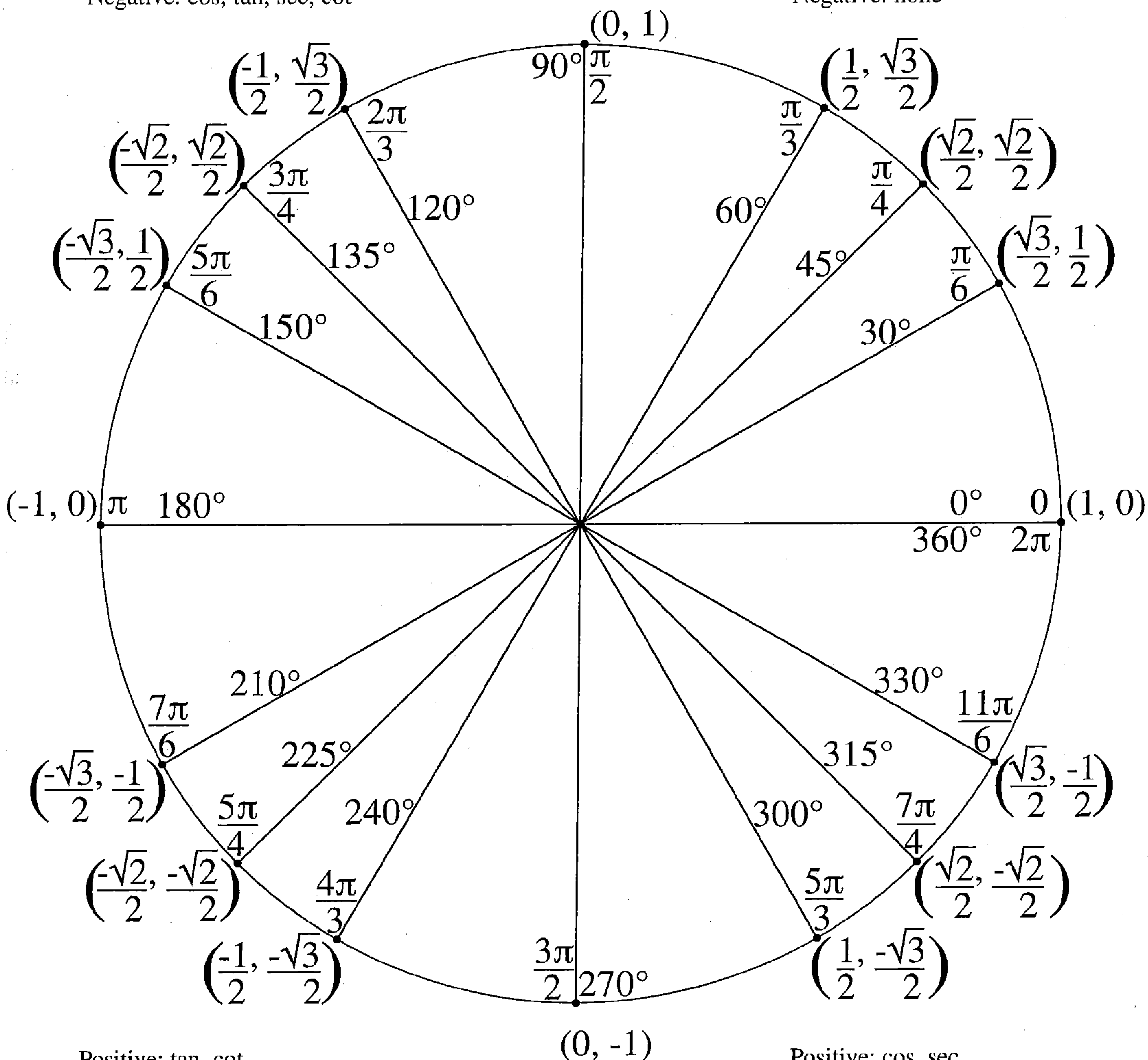
IV

The Unit Circle

Positive: sin, csc
Negative: cos, tan, sec, cot

Positive: sin, cos, tan, sec, csc, cot
Negative: none



Positive: tan, cot
Negative: sin, cos, sec, csc

Positive: cos, sec
Negative: sin, tan, csc, cot

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V

Review of Complex Numbers

Introduction

This is a short review of the main concepts of *complex numbers.* Complex numbers are used throughout mathematics and its applications. In particular, when we try to solve differential equations it is often convenient and natural to use complex numbers to express the solutions. Here we review those ideas and results from the theory of complex numbers that will be used in Math 216.

A complex number z may be expressed as an ordered pair of *real* numbers:

$$z = (x, y) = x + iy$$

where $i := \sqrt{-1}$ (so $i^2 = -1$) and x and y are real numbers. The following notations are often used:

$$\begin{aligned} x &= \mathrm{Re}(z) \text{ or } x = \Re(z) \text{ denotes the } \textit{real part} \text{ of the complex number } z \\ y &= \mathrm{Im}(z) \text{ or } y = \Im(z) \text{ denotes the } \textit{imaginary part} \text{ of the complex number } z \end{aligned}$$

Recall that two complex numbers are equal if and only if both the real and the imaginary parts are equal. In other words, $z_1 := (x_1, y_1)$ equals $z_2 := (x_2, y_2)$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

A convenient way of thinking about complex numbers is to imagine them as points in the (x, y) plane (in this case it is called the *complex plane*), as illustrated in the following figure. In the complex plane, the line $y = 0$ is frequently called the *real axis*, and the line

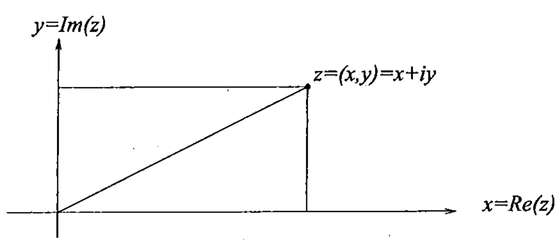

Figure 1: The complex number $z = x + iy$ plotted in the complex plane.

$x = 0$ is frequently called the *imaginary axis.*

Doing arithmetic with complex numbers

Addition and multiplication of two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are defined by the following rules:

- Addition: $z_1 + z_2 := (x_1 + x_2, y_1 + y_2) = (x_1 + x_2) + i(y_1 + y_2)$.
- Multiplication: $z_1 z_2 := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

Note that if we interpret z_1 and z_2 as points in the complex plane as in Figure 1, then addition of complex numbers is the same as vector addition in the plane; we are just adding the real and imaginary parts componentwise. On the other hand, the multiplication of two complex numbers may perhaps seem different than what you might have expected it to be; this is only an illusion, however, and when we introduce exponential forms for complex numbers later, the multiplication will make perfect sense.

Although complex numbers obey different rules of arithmetic than do ordinary real numbers, it is very important to keep in mind that the complex numbers simply generalize the notion of the real numbers. Indeed, we can think of the real number x as the complex number $(x, 0) = x + i0$. Such a complex number whose imaginary part is zero is said to be *purely real.* If we add or multiply two purely real complex numbers, then according to the rules for complex arithmetic, we have

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \quad \text{and} \quad (x_1, 0)(x_2, 0) = (x_1x_2, 0)$$

so in each case the result is also a purely real complex number, and the real part in each case is exactly what we would have found by applying the usual rules of addition and multiplication for real numbers to the real parts. This shows that all the new operations defined for complex numbers when applied to purely real numbers give the usual familiar corresponding operations.

One way to think of $(0, 1)$ is as the *new* number i which is *purely imaginary* in the sense that its real part is zero, and so $(x, y) = x + iy$ is the sum of the purely real number x and the purely imaginary number iy.

Example: According to the above arithmetic rules for complex arithmetic, we have

$$(x, 0) + (0, y) = (x, y)\,, \quad \text{and} \quad (0, 1)(y, 0) = (0, y)\,.$$

Combining these, we deduce that

$$(x, y) = (x, 0) + (0, 1)(y, 0)$$

which is another way of writing the relation $z = x + iy$. $\square$

Example: We can calculated repeated products of a complex number z with itself, which is what we mean by raising z to an integer power. Thus by definition really, $z^2 = zz$ and $z^3 = zzz$ and so on. In particular, $i^2 = ii = (0, 1)(0, 1)$. Using the rule for multiplication, we then see that

$$i^2 = ii = (0, 1)(0, 1) = (-1, 0) = -1$$

which verifies the fact that i is a square root of -1. $\square$

Example: The fact that $i^2 = -1$ makes the rule for multiplication of complex numbers very easy to remember if one uses the $z = x + iy$ notation. Indeed just by multiplying out the individual terms,

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2$$

and then using $i^2 = -1$ we get

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

which is the rule for multiplication of complex numbers. $\square$

It is easy to check directly from the definitions given of addition and multiplication of complex numbers that all of the familiar algebraic properties that we are familiar with hold for complex numbers too. In other words, complex arithmetic obeys the following rules:

- Commutative Law of Addition: $z_1 + z_2 = z_2 + z_1$
- Associative Law of Addition: $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- Commutative Law of Multiplication: $z_1 z_2 = z_2 z_1$
- Distributive Law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
- Unique Additive Identity $0 = (0,0) : z + 0 = 0 + z = z$
- Unique Multiplicative Identity $1 = (1,0) : z \cdot 1 = 1 \cdot z = z$
- Additive inverse: $-z = (-x, -y) = -x - iy : z + (-z) = 0$
- Multiplicative inverse: For every complex number $z = (x,y) \neq 0$ there exists a complex number $w = (u,v)$ such that $(x,y)(u,v) = (u,v)(x,y) = (1,0)$

It turns out that the multiplicative inverse of a nonzero complex number $z = (x,y)$ is the complex number

$$\left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right)$$

which we denote by $1/z$. Now we can define the quotient of two complex numbers:

$$\frac{z_1}{z_2} := z_1 \cdot \frac{1}{z_2} = \frac{1}{z_2} \cdot z_1 \, .$$

Example: The multiplicative inverse of $i = (0,1)$ is, according to the above formula,

$$\frac{1}{i} = (0,-1) = -i \, .$$

Therefore,

$$\frac{2}{i} = 2 \cdot \frac{1}{i} = -2i \, .$$

As a more complicated example, since

$$\frac{1}{1+i} = \left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}i \, ,$$

we have

$$\frac{2-3i}{1+i} = (2-3i)\cdot\left(\frac{1}{2}-\frac{1}{2}i\right) = 1-i-\frac{3}{2}i+\frac{3}{2}i^2 = -\frac{1}{2}-\frac{5}{2}i$$

because $i^2 = -1$. □

Why bother with complex numbers at all? Complex numbers were originally invented as an extension of real numbers in order to have a number system in which all polynomials have roots. For example, the equation $x^2 - 3x + 2 = 0$ has two real solutions, $x = 1$ or $x = 2$. But the similar-looking quadratic equation $x^2 - 3x + 3 = 0$ does not have any real roots at all! However, if we are willing to accept complex numbers as roots, then this quadratic equation also has two roots, namely the complex roots $3/2+i\sqrt{3}/2$ and $3/2-i\sqrt{3}/2$. More generally, for $a_n \neq 0$, and other given numbers $a_0, \ldots, a_{n-1}$, the polynomial equation $a_n x^n + ... + a_1 x_1 + a_0 = 0$ of degree n does not necessarily have any real solutions. However, complex numbers enjoy the property that if $a_0, a_1, ..., a_n$ are complex numbers and $a_n \neq 0$, then $a_n z^n + ... + a_1 z_1 + a_0 = 0$ always has n solutions (although not all roots are necessarily distinct). This fact is known as the *Fundamental Theorem of Algebra.* In other words, to find the solutions of a polynomial equation, you never need to look further than the complex numbers (remarkably, this is true even if the coefficients a_k are themselves generalized from real numbers to complex numbers). *This is exactly why we need complex numbers in a course on differential equations like Math 216: they are necessary to give us all of the roots of the characteristic polynomial that arises from seeking exponential solutions proportional to e^{rt} of a constant-coefficient differential equation, or system of differential equations.*

Some additional terminology for complex numbers

Associated with each complex number z is a positive number called the *absolute value* or *modulus* of z and written as $|z|$; the definition in terms of the real and imaginary parts of $z = x + iy$ is

$$|z| := \sqrt{x^2 + y^2}\,.$$

If we visualize z as a point in the complex plane as in Figure 1, the modulus of z is just the distance from the point (x, y) to the origin $(0, 0)$. See Figure 2. Unless z_1 and z_2

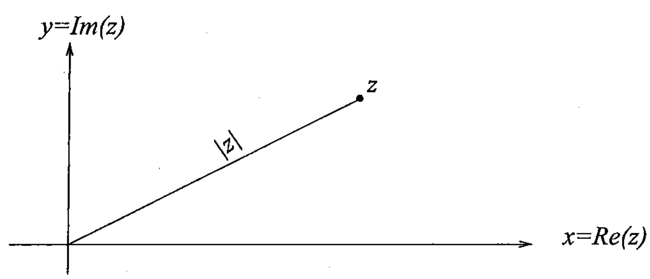

Figure 2: The modulus of a complex number $z = x + iy$.

are purely real, an inequality like "$z_1 < z_2$" has no meaning because somehow both real

and imaginary parts would have to be compared. But the inequality $|z_1| < |z_2|$ does have meaning; according to Figure 2 it means that z_1 is closer to $(0,0)$ than z_2 is.

Example: According to the definition, $|(1,2)| = \sqrt{1^2+2^2} = \sqrt{5}$. □

Next, for each complex number z, there is another complex number called the *complex conjugate* of z and denoted by $\overline{z}$ or z^*. The complex conjugate of $z = (x,y) = x+iy$ is defined by

$$\overline{z} = z^* := (x,-y) = x - iy\,,$$

so taking the complex conjugate of a complex number z amounts to changing the sign of its imaginary part. Geometrically, this amounts to reflection of the point representing z in the complex plane through the real axis, as shown in Figure 3.

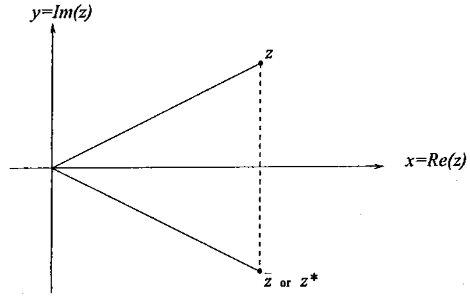

Figure 3: The complex conjugate of a complex number z.

Example: According to the definition, $\overline{1} = \overline{(1,0)} = (1,0) = 1$. Similarly, $i^* = (0,1)^* = (0,-1) = -i$. □

Example: The following identities are easy to establish using the definition:

$$\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}\,,$$

$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}\,,$$

and

$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}\,.$$

Also, $\overline{\overline{z}} = z$, in other words, the complex conjugate of the complex conjugate of any number z is z itself. Why are there two different notations for the complex conjugate of a complex number? Generally, the bar notation is easier to read, unless it gets in the way of a dot on the i, or unless it is easily confused with the line separating the numerator and denominator of a fraction; for these situations, we have the option of using the star superscript. □

Example: Note that if $z = (x,y) = x+iy$, then

$$z + \overline{z} = (x,y) + (x,-y) = (2x,0) = 2 \cdot \mathrm{Re}(z)\,,$$

$$z - \overline{z} = (x, y) - (x, -y) = (0, 2y) = 2i \cdot \mathrm{Im}(z)\,.$$

That is, we have that

$$\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \quad \text{and} \quad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})\,.$$

This gives us a simple way to express the real and imaginary parts of z in terms of z and its complex conjugate. □

Example: Again, directly from the definition of complex conjugation,

$$zz^* = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2 = |z|^2\,,$$

where in the last step we used the definition of the modulus of z. It follows that by multiplying the numerator and denominator of $1/z$ by z^*, we get

$$z^{-1} = \frac{1}{z} = \frac{z^*}{zz^*} = \frac{z^*}{|z|^2}\,.$$

This is one way to deduce the formula we gave earlier for $1/z$. □

Polar form for complex numbers

Points in the complex plane can be identified by their Cartesian coordinates (x, y), or by their polar coordinates (r, θ) as indicated in Figure 4. Elementary trigonometry tells us that

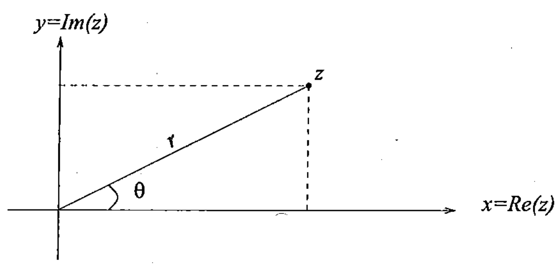

Figure 4: The polar coordinates of the complex number z.

the Cartesian and polar coordinates are related by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. We may therefore write the complex number z in terms of its polar coordinates in the following way:

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))\,.$$

Geometrically, it is easy to see that the modulus $|z|$ of z is the same thing as the polar coordinate r; however it is also easy to see this from the above formula using the trigonometry identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$:

$$|z| = \sqrt{r^2\cos(\theta)^2 + r^2\sin(\theta)^2} = r\sqrt{\cos^2\theta + \sin^2\theta} = r\,.$$

Example: For $z = 1 - i$ we have $r = \sqrt{2}$ and $\theta = -\pi/4$, and therefore

$$1 - i = \sqrt{2}[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})] .$$

The angle θ is not unique, but its possible values differ from each other by multiples of 2π. For example, $\theta = 2\pi n - \pi/4$ works too, for any $n = 0, \pm 1, \pm 2, \ldots$. □

The angle θ is called the *argument* or *phase angle* of z and is denoted

$$\theta = \arg(z) .$$

Again thinking geometrically, it is easy to see that for a complex number to have an argument, it must be nonzero, which is the same thing as saying that its modulus is nonzero, or that $r \neq 0$. Since for $z \neq 0$ there are many values of $\arg(z)$, it is useful to define a particular value, called the *principal value* of $\arg(z)$ and denoted by $\mathrm{Arg}(z)$ as the unique value of θ between $-\pi$ and π:

$$-\pi < \mathrm{Arg}(z) \leq \pi .$$

If we are given the polar coordinates (r, θ) of a complex number z, then it is straightforward to calculate the corresponding Cartesian coordinates $(x, y) = (r\cos(\theta), r\sin(\theta))$. Finding the polar coordinates given the Cartesian coordinates is a little more tricky. Of course it is easy to find $r = \sqrt{x^2 + y^2}$, but then we need to find an angle θ so that

$$\cos(\theta) = \frac{x}{r}, \quad \text{and} \quad \sin(\theta) = \frac{y}{r} .$$

It also follows from these relations that the angle θ we seek satisfies

$$\tan(\theta) = \frac{y}{x} = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} .$$

The way to find θ is to use the inverse trigonometric functions; however in doing so you need to make sure that the resulting angle is in the correct quadrant given the signs of x and y.

Example: Suppose that $z = x + iy = -1 + i$. To find the polar coordinates of z, we first calculate $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$. Then, the angle θ we seek satisfies $\cos(\theta) = -1/\sqrt{2}$ and $\sin(\theta) = 1/\sqrt{2}$, or if we combine these, $\tan(\theta) = -1$. If we apply the arctangent function to solve for θ, we get $-\pi/4$, which is indeed an angle whose tangent is -1. However it is not an angle whose sine is $1/\sqrt{2}$. To get the right answer we recall that the arctangent function is only defined up to integer multiples of π, and therefore $\theta = 3\pi/4$ is also an angle whose tangent is -1. In this case, we also see that indeed $\cos(3\pi/4) = -1/\sqrt{2}$ and that $\sin(3\pi/4) = 1/\sqrt{2}$, so that indeed $\theta = \arg(z) = 3\pi/4$. Moreover, since $-\pi < 3\pi/4 \leq \pi$, we also have $\mathrm{Arg}(z) = 3\pi/4$. □

One of the reasons for introducing polar coordinates for complex numbers is that it gives a simple geometrical interpretation to the process of complex multiplication. Indeed, if

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) , \quad \text{and} \quad z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2)) ,$$

then

$$\begin{aligned}
z_1 z_2 &= r_1 r_2 \left(\cos(\theta_1) + i\sin(\theta_1)\right)\left(\cos(\theta_2) + i\sin(\theta_2)\right) \\
&= r_1 r_2 \left[\left(\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)\right) + i\left(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)\right)\right] \\
&= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] .
\end{aligned}$$

We used two trigonometric identities in the last step above to arrive at a formula in terms of the sum of the angles $\theta_1 + \theta_2$. In particular, this calculation shows that

$$|z_1 z_2| = |z_1| \cdot |z_2| , \qquad \text{and} \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) .$$

That is, when we multiply two complex numbers we multiply their moduli and add their arguments. Note also that if $r \neq 0$, then

$$\frac{1}{z} = \frac{1}{r(\cos(\theta) + i\sin(\theta))} \cdot \frac{r(\cos(\theta) - i\sin(\theta))}{r(\cos(\theta) - i\sin(\theta))} = \frac{1}{r}(\cos(\theta) - i\sin(\theta)) = \frac{1}{r}\left[\cos(-\theta) + i\sin(-\theta)\right] ,$$

so $|z^{-1}| = 1/|z|$ and $\arg(z^{-1}) = -\arg(z)$. Note also that $\arg(\overline{z}) = -\arg(z)$.

Exponential form of a complex number: Euler's formula

Recall that the infinite power series expansion defining e^w is

$$e^w = 1 + w + \frac{w^2}{2} + \frac{w^3}{3!} + \cdots .$$

It turns out that both sides make sense when w is a complex number, and in particular if $w = i\theta$ where θ is a real angle. In this case the series expansion becomes

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \cdots \\
&= \left[1 - \frac{\theta^2}{2} + \cdots\right] + i\left[\theta - \frac{\theta^3}{3!} + \cdots\right] \\
&= \cos(\theta) + i\sin(\theta) ,
\end{aligned}$$

where at the end we grouped the purely real and purely imaginary terms and recalled the infinite power series expansions of $\cos(\theta)$ and $\sin(\theta)$ for real values of θ. The remarkable formula we have found in this way:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is known as *Euler's formula*. If we use Euler's formula, we can express z in terms of its polar coordinates in an even simpler form:

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta} ,$$

where $r = |z|$ is the absolute value of z and $\theta = \arg(z)$ is the argument. This is called the *exponential form* of the complex number z.

Now we can see that we can also view the effect of multiplying two complex numbers that are expressed in terms of their polar coordinates (multiplication of the moduli and addition of the arguments) as being a simple consequence of the rules for multiplying exponentials:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} .$$

Also,

$$\frac{1}{z} = \frac{1}{re^{i\theta}} = \left(\frac{1}{r}\right) e^{-i\theta} ,$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} .$$

Note that for any $n = 0, \pm 1, \pm 2, \ldots$ we have $z = re^{i(\theta+2\pi n)}$; also $\overline{z} = re^{-i\theta}$.

The exponential form of a complex number z makes it easy to compute powers of z. Indeed, $z^n = (re^{i\theta})^n = r^n e^{in\theta}$. Combining this with Euler's formula we have *DeMoivre's Theorem*, which gives the real and imaginary parts of any power of a complex number of modulus one:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta) .$$

The proof of DeMoivre's theorem is basically one line long:

$$(\cos(\theta) + i\sin(\theta))^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta) .$$

Euler's formula was used in the first and last steps.

Example: Writing out DeMoivre's formula for $n = 2$, and doing the multiplication, gives

$$\cos(2\theta) + i\sin(2\theta) = (\cos(\theta) + i\sin(\theta))^2 = \cos(\theta)^2 - \sin(\theta)^2 + i2\sin(\theta)\cos(\theta) .$$

Since the real and imaginary parts on both sides must be equal, we get $\cos(2\theta) = \cos(\theta)^2 - \sin(\theta)^2$ and $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. This example shows that DeMoivre's Theorem provides an easy way to remember the multiple-angle trigonometry identities. □

Finding roots of complex numbers is also made easy using the exponential form. For example,

$$z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2} ,$$

but, since θ may be replaced by $\theta + 2\pi k$ for any integer k without changing z, we have more generally that

$$(re^{i(\theta+2\pi k)})^{1/2} = r^{1/2} e^{i(\theta/2+\pi k)} .$$

The right-hand side gives only two possible answers as k ranges over all possible integers. These two complex numbers are the two square roots of z. The nth roots of a complex number z are calculated in exactly the same way:

$$\begin{aligned} z^{1/n} &= r^{1/n} e^{i(\theta+2\pi k)/n} \\ &= r^{1/n} e^{i(\theta/n+2\pi k/n)} , \end{aligned}$$

VI

Introduction to Matrices

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1 Definitions

A matrix (plural: matrices) is simply a rectangular array of "things". For now, we'll assume the "things" are numbers, but as you go on in mathematics, you'll find that matrices can be arrays of very general objects. Pretty much all that's required is that you be able to add, subtract, and multiply the "things".

Here are some examples of matrices. Notice that it is sometimes useful to have variables as entries, as long as the variables represent the same sorts of "things" as appear in the other slots. In our examples, we'll always assume that all the slots are filled with numbers. All our examples contain only real numbers, but matrices of complex numbers are very common.

$$\begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 4 & x & 17 \\ 2 & x+y & 7 & -19 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} x & y & z & w \end{pmatrix}$$

The first example is a square 3×3 matrix; the next is a 2×4 matrix (2 rows and 4 columns—if we talk about a matrix that is "$m \times n$" we mean it has m rows and n columns). The final two examples consist of a single column matrix, and a single row matrix. These final two examples are often called "vectors"—the first is called a "column vector" and the second, a "row vector". We'll use only column vectors in this introduction.

Often we are interested in representing a general $m \times n$ matrix with variables in every location, and that is usually done as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

The number in row i and column j is represented by a_{ij}, where $1 \le i \le m$ and $1 \le j \le n$. Sometimes when there is no question about the dimensions of a matrix, the entire matrix can simply be referred to as:

$$(a_{ij}).$$

1.1 Addition and Subtraction of Matrices

As long as you can add and subtract the "things" in your matrices, you can add and subtract the matrices themselves. The addition and subtraction occurs in the obvious way—element by element. Here are a couple of examples:

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 6 & -4 \\ 2 & 15 & \pi \end{pmatrix} + \begin{pmatrix} 3 & 2 & 1 \\ 5.5 & 3 & -e \\ 2 & 5 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 4 & 5 & 8 \\ 7.5 & 9 & -4-e \\ 4 & 20 & \pi+\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 6 & -4 \\ 2 & 15 & \pi \end{pmatrix} - \begin{pmatrix} 3 & 2 & 1 \\ 5.5 & 3 & -e \\ 2 & 5 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 6 \\ -3.5 & 3 & e-4 \\ 0 & 10 & \pi - \sqrt{2} \end{pmatrix}$$

To find what goes in row i and column j of the sum or difference, just add or subtract the entries in row i and column j of the matrices being added or subtracted.

In order to make sense, both of the matrices in the sum or difference must have the same number of rows and columns. It makes no sense, for example, to add a 2×4 matrix to a 3×4 matrix.

1.2 Multiplication of Matrices

When you add or subtract matrices, the two matrices that you add or subtract must have the same number of rows and the same number of columns. In other words, both must have the same shape.

For matrix multiplication, all that is required is that the number of columns of the first matrix be the same as the number of rows of the second matrix. In other words, you can multiply an $m \times k$ matrix by a $k \times n$ matrix, with the $m \times k$ matrix on the left and the $k \times n$ matrix on the right. The example on the left below represents a legal multiplication since there are three columns in the left multiplicand and three rows in the right one; the example on the right doesn't make sense—the left matrix has three columns, but the right one has only 2 rows. If the matrices on the right were written in the reverse order with the 2×3 matrix on the left, it would represent a valid matrix multiplication.

$$\begin{pmatrix} 1 & 3 & 5 \\ 4 & 7 & 2 \\ 9 & 1 & 6 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 3 & 7 \\ 7 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

So now we know what shapes of matrices it is legal to multiply, but how do we do the actual multiplication? Here is the method:

If we are multiplying an $m \times k$ matrix by a $k \times n$ matrix, the result will be an $m \times n$ matrix. The element in the product in row i and column j is gotten by multiplying term-wise all the elements in row i of the matrix on the left by all the elements in column j of the matrix on the right and adding them together.

Here is an example:

$$\begin{pmatrix} 1 & 3 & 2 \\ 5 & 0 & 7 \\ 6 & 9 & 8 \end{pmatrix} \begin{pmatrix} 4 & 11 \\ 6 & 10 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} 32 & 59 \\ 55 & 118 \\ 118 & 228 \end{pmatrix}$$

To find what goes in the first row and first column of the product, take the number from the first row of the matrix on the left: $(1, 3, 2)$, and multiply them, in order, by the numbers in the first column of the matrix on the right: $(4, 6, 5)$. Add the results: $1 \cdot 4 + 3 \cdot 6 + 2 \cdot 5 = 4 + 18 + 10 = 32$. To get the 228 in the third row and second column of the product, the use the numbers in the third row of the left matrix: $(6, 9, 8)$ and the numbers in the second column of the right matrix: $(11, 10, 9)$ to get $6 \cdot 11 + 9 \cdot 10 + 8 \cdot 9 = 66 + 90 + 72 = 228$.

Check your understanding by verifying that the other elements in the product matrix are correct.

In general, if we multiply a general $m \times k$ matrix by a general $k \times n$ matrix to get an $m \times n$ matrix as follows:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kn} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{pmatrix}$$

Then we can write c_{ij} (the number in row i, column j) as:

$$c_{ij} = \sum_{p=1}^{k} a_{ip} b_{pj}.$$

1.3 Square Matrices and Column Vectors

Although everything above has been stated in terms of general rectangular matrices, for the rest of this tutorial, we'll consider only two kinds of matrices (but of any dimension): square matrices, where the number of rows is equal to the number of columns, and column matrices, where there is only one column. These column matrices are often called "vectors", and there are many applications where they correspond exactly to what you commonly use as sets of coordinates for points in space. In the two-dimensional x-y plane, the coordinates $(1,3)$ represent a point that is one unit to the right of the origin (in the direction of the x-axis), and three units above the origin (in the direction of the y-axis). That same point can be written as the following column vector:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

If you wish to work in three dimensions, you'll need three coordinates to locate a point relative to the (three-dimensional) origin—an x-coordinate, a y-coordinate, and a z-coordinate. So the point you'd normally write as (x, y, z) can be represented by the column vector:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Quite often we will work with a combination of square matrices and column matrices, and in that case, if the square matrix has dimensions $n \times n$, the column vectors will have dimension $n \times 1$ (n rows and 1 column)[1].

1.4 Properties of Matrix Arithmetic

Matrix arithmetic (matrix addition, subtraction, and multiplication) satisfies many, *but not all* of the properties of normal arithmetic that you are used to. All of the properties below can be formally proved, and it's not too difficult, but we will not do so here. In what follows, we'll assume that different matrices are represented by upper-case letters: $M, N, P, \ldots$, and that column vectors are represented by lower-case letters: $v, w, \ldots$.

We will further assume that all the matrices are square matrices or column vectors, and that all are the same size, either $n \times n$ or $n \times 1$. Further, we'll assume that the matrices contain numbers (real or complex). Most of the properties listed below apply equally well to non-square matrices, assuming that the dimensions make the various multiplications and addtions/subtractions valid.

Perhaps the first thing to notice is that we can always multiply two $n \times n$ matrices, and we can multiply an $n \times n$ matrix by a column vector, but we cannot multiply a column vector by the matrix, nor a column vector by another. In other words, of the three matrix multiplications below, only the first one makes sense. Be sure you understand why.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \qquad \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

[1] We could equally well use row vectors to correspond to coordinates, and this convention is used in many places. However, the use of column matrices for vectors is more common

Finally, an extremely useful matrix is called the "identity matrix", and it is a square matrix that is filled with zeroes except for ones in the diagonal elements (having the same row and column number). Here, for example, is the 4×4 identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The identity matrix is usually called "I" for any size square matrix. Usually you can tell the dimensions of the identity matrix from the surrounding context.

- Associative laws:

$$\begin{array}{ll} (MN)P = M(NP) & (MN)v = M(Nv) \\ (M+N)+P = M+(N+P) & (u+v)+w = u+(v+w) \end{array}$$

- Commutative laws for addition:

$$\begin{array}{ll} M+N = N+M & v+w = w+v \end{array}$$

- Distributive laws:

$$\begin{array}{ll} M(N \pm P) = MN \pm MP & (M \pm N)P = MP \pm NP \\ M(v \pm w) = Mv \pm Mw & (M \pm N)v = Mv \pm Nv \end{array}$$

- The identity matrix:

$$\begin{array}{ll} NI = IN = N & Iv = v \end{array}$$

Probably the most important thing to notice about the laws above is one that's missing—multiplication of matrices is not in general commutative. It is easy to find examples of matrices M and N where $MN \neq NM$. In fact, matrices almost never commute under multiplication. Here's an example of a pair that don't:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

So the order of multiplication is very important; that's why you may have noticed the care that has been taken so far in describing multiplication of matrices in terms of "the matrix on the left", and "the matrix on the right".

The associative laws above are extremely useful, and to take one simple example, consider computer graphics. As we'll see later, operations like rotation, translation, scaling, perspective, and so on, can all be represented by a matrix multiplication. Suppose you wish to rotate all the vectors in your drawing and then to translate the results. Suppose R and T are the rotation and translation matrices that do these jobs. If your picture has a million points in it, you can take each of those million points v and rotate them, calculating Rv for each vector v. Then, the result of that rotation can be translated: $T(Rv)$, so in total, there are two million matrix multiplications to make your picture. But the associative law tells us we can just multiply T by R once to get the matrix TR, and then multiply all million points by TR to get $(TR)v$, so all in all, there are only 1,000,001 matrix multiplications— one to produce TR and a million multiplications of TR by the individual vectors. That's quite a savings of time.

The other thing to notice is that the identity matrix behaves just like 1 under multiplication—if you multiply any number by 1, it is unchanged; if you multiply any matrix by the identity matrix, it is unchanged.