55. $3m^3 + 12m^2 + 9m$

56. $9y^4 - 54y^3 + 45y^2$

57. $6k^2 + 5kp - 6p^2$

58. $14m^2 + 11mr - 15r^2$

59. $5a^2 - 7ab - 6b^2$

60. $12s^2 + 11st - 5t^2$

61. $21x^2 - xy - 2y^2$

62. $30a^2 + am - m^2$

63. $24a^4 + 10a^3b - 4a^2b^2$ $2a^2(4a - b)(3a + 2b)$

64. $18x^5 + 15x^4z - 75x^3z^2$ $3x^3(3x - 5z)(2x + 5z)$

65. $15x^2y^5 - 20x^3y^3 + 15xy^2$ $5xy^2(3xy^3 - 4x^2y + 3)$

66. $28m^4n^6 + 21m^6n^3 - 35m^3n^2$ $7m^3n^2(4mn^4 + 3m^3n - 5)$

Factor each perfect square trinomial. It may be necessary to factor out a common factor first.

67. $9m^2 - 12m + 4$

68. $16p^2 - 40p + 25$

69. $32a^2 - 48ab + 18b^2$

70. $20p^2 - 100pq + 125q^2$

71. $4x^2y^2 + 28xy + 49$

72. $9m^2n^2 - 12mn + 4$

Factor each difference of squares.

73. $x^2 - 36$

74. $t^2 - 64$

75. $y^2 - w^2$

76. $25 - w^2$

77. $9a^2 - 16$

78. $16q^2 - 25$

79. $25s^4 - 9t^2$

80. $36z^2 - 81y^4$

81. $p^4 - 625$

82. $m^4 - 81$

Factor each sum or difference of cubes.

83. $8 - a^3$

84. $r^3 + 27$

85. $125x^3 - 27$

86. $8m^3 - 27n^3$

87. $27y^9 + 125z^6$

88. $27z^3 + 729y^3$

Decide on a factoring method, and then factor the polynomial completely.

89. $x^2 + xy - 5x - 5y$

90. $8r^2 - 10rs - 3s^2$

91. $12m^2 + 16mn - 35n^2$

92. $36a^2 + 60a + 25$

93. $4z^2 + 28z + 49$

94. $6p^4 + 7p^2 - 3$

95. $1000x^3 + 343y^3$

96. $b^2 + 8b + 16 - a^2$

97. $125m^6 - 216$

98. $q^2 + 6q + 9 - p^2$

99. $p^4(m - 2n) + q(m - 2n)$ $(m - 2n)(p^4 + q)$

100. $216p^3 + 125q^3$ $(6p + 5q)(36p^2 - 30pq + 25q^2)$

7.7 QUADRATIC EQUATIONS AND APPLICATIONS

Quadratic Equations • Zero-Factor Property • Square Root Property • Quadratic Formula • Applications

Quadratic Equations

Quadratic Equation

An equation that can be written in the form

$$ax^2 + bx + c = 0 \quad \text{Standard form}$$

where a, b, and c are real numbers, with $a \neq 0$, is a **quadratic equation.** The form of the equation given above is called **standard form.**

Zero-Factor Property

The simplest method of solving a quadratic equation, but one that is not always easily applied, is by factoring. This method depends on the following property.

Zero-Factor Property

If $ab = 0$, then $a = 0$ or $b = 0$ or both.

When solving a quadratic equation by the zero-factor property, the equation must be in standard form before factoring.

EXAMPLE 1 Using the Zero-Factor Property

Solve $6x^2 + 7x = 3$.

SOLUTION

$$6x^2 + 7x = 3$$

$$6x^2 + 7x - 3 = 0 \qquad \text{Standard form}$$

$$(3x - 1)(2x + 3) = 0 \qquad \text{Factor.}$$

$$3x - 1 = 0 \quad \text{or} \quad 2x + 3 = 0 \qquad \text{Zero-factor property}$$

$$3x = 1 \quad \text{or} \quad 2x = -3 \qquad \text{Solve each equation.}$$

$$x = \frac{1}{3} \quad \text{or} \quad x = -\frac{3}{2} \qquad \text{Divide.}$$

Check by first substituting $\frac{1}{3}$ and then $-\frac{3}{2}$ in the original equation. The solution set is $\left\{\frac{1}{3}, -\frac{3}{2}\right\}$.

Square Root Property

A quadratic equation of the form $x^2 = k$, $k \geq 0$, can be solved by factoring.

$$x^2 = k$$

$$x^2 - k = 0 \qquad \text{Subtract } k.$$

$$(x + \sqrt{k})(x - \sqrt{k}) = 0 \qquad \text{Factor, using radicals.}$$

$$x + \sqrt{k} = 0 \quad \text{or} \quad x - \sqrt{k} = 0 \qquad \text{Zero-factor property}$$

$$x = -\sqrt{k} \quad \text{or} \quad x = \sqrt{k} \qquad \text{Solve each equation.}$$

This leads to the square root property for solving equations.

Square Root Property

If $k \geq 0$, then the solutions of $x^2 = k$ are $x = \pm\sqrt{k}$.

If $k > 0$, the equation $x^2 = k$ has two real solutions. If $k = 0$, there is only one solution, 0. If $k < 0$, there are no real solutions. (However, in this case, there *are* imaginary solutions. Imaginary numbers are discussed briefly in the **Extensions** on complex numbers at the end of **Chapter 6** and this chapter.)

EXAMPLE 2 Using the Square Root Property

Use the square root property to solve each quadratic equation for real solutions.

(a) $x^2 = 25$ **(b)** $x^2 = 18$ **(c)** $x^2 = -3$ **(d)** $(x - 4)^2 = 12$

SOLUTION

(a) Since $\sqrt{25} = 5$, the solution set of the equation $x^2 = 25$ is $\{5, -5\}$, which may be abbreviated $\{\pm 5\}$.

Completing the square, used in deriving the quadratic formula, has important applications in algebra. To transform the expression $x^2 + kx$ into the square of a binomial, we add to it the square of half the coefficient of x, that is,

$$\left[\left(\tfrac{1}{2}\right)k\right]^2 = \tfrac{k^2}{4}.$$

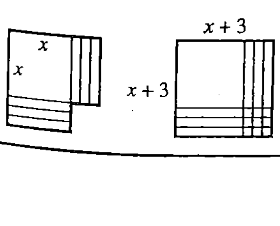
We then get

$$x^2 + kx + \frac{k^2}{4} = \left(x + \frac{k}{2}\right)^2.$$

For example, to make $x^2 + 6x$ the square of a binomial, we add 9, since $9 = \left[\frac{1}{2}(6)\right]^2$. This results in the trinomial $x^2 + 6x + 9$, which is equal to $(x + 3)^2$.

The Greeks had a method of completing the square geometrically. For example, to complete the square for $x^2 + 6x$, begin with a square of side x. Add three rectangles of width 1 and length x to the right side and the bottom. Each rectangle has area $1x$ or x, so the total area of the figure is now $x^2 + 6x$. To fill in the corner (that is, "complete the square"), we add 9 1-by-1 squares as shown. The new completed square has sides of length $x + 3$ and area

$$(x + 3)^2 = x^2 + 6x + 9.$$



(b)

$$
\begin{aligned}
x^2 &= 18 \\
x &= \pm\sqrt{18} && \text{Square root property} \\
x &= \pm\sqrt{9 \cdot 2} && \text{Factor 18 as } 9 \cdot 2. \\
x &= \pm\sqrt{9} \cdot \sqrt{2} && \text{Product rule for square roots} \\
x &= \pm 3\sqrt{2} && \sqrt{9} = 3
\end{aligned}
$$

The solution set is $\{\pm 3\sqrt{2}\}$.

(c) Since $-3 < 0$, the equation $x^2 = -3$ has no real solutions. The solution set is $\emptyset$.

(d)

$$
\begin{aligned}
(x-4)^2 &= 12 \\
x - 4 &= \pm\sqrt{12} && \text{Square root property} \\
x &= 4 \pm \sqrt{12} && \text{Add 4.} \\
x &= 4 \pm \sqrt{4 \cdot 3} && \text{Factor 12 as } 4 \cdot 3. \\
x &= 4 \pm 2\sqrt{3} && \sqrt{4 \cdot 3} = 2\sqrt{3}
\end{aligned}
$$

The solution set is $\{4 \pm 2\sqrt{3}\}$.

Quadratic Formula

By *completing the square* (see the margin note on **page 349**) we can derive one of the most important formulas in algebra, the *quadratic formula.*

$$
\begin{aligned}
ax^2 + bx + c &= 0 && \text{Standard quadratic equation } (a > 0) \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{Divide by } a. \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} && \text{Add } -\frac{c}{a}. \\
x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} &= \frac{b^2}{4a^2} - \frac{c}{a} && \text{Add } \frac{b^2}{4a^2}. \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && \text{Factor on the left. Combine terms on the right.} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} && \text{Square root property} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} && \text{Quotient rule for square roots} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} && \text{Subtract } \frac{b}{2a}. \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Combine like terms. This is also valid for } a < 0.
\end{aligned}
$$

Be careful; $-b \pm \sqrt{b^2 - 4ac}$ **is all written over 2a.**

A first-season episode of *Blue Collar TV* (2004) featured Bill Engvall paying a sarcastic tribute to an underappreciated figure in his life.

To my high school algebra teacher, for teaching me that x equals minus b plus or minus the square root of b squared minus 4ac all over 2a, because Lord knows I use that information EVERY DAY!

Bill's teacher evidently did a good job, because he used the word "all" before $2a$. A common student error is to forget to write the $-b$ in the numerator with the radical expression in the **quadratic formula.** See **Exercise 42** in this section.

Quadratic Formula

The solutions of $ax^2 + bx + c = 0, a \neq 0$, are given by the **quadratic formula.**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A Radical Departure from the Other Methods of Evaluating the Golden Ratio Recall from a previous chapter that the golden ratio is found in numerous places in mathematics, art, and nature. In a margin note there, we showed that *the "continued" fraction*

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}$$

is equal to the golden ratio, $\frac{1+\sqrt{5}}{2}$. Now consider this *"nested" radical:*

$$\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}$$

Let x represent this radical expression. Because it appears "within itself," we can write

$$x=\sqrt{1+x}$$
$$x^2=1+x$$
$$x^2-x-1=0.$$

Using the quadratic formula, with $a=1$, $b=-1$, and $c=-1$, it can be shown that the positive solution of this equation, and thus the value of the nested radical is . . . (you guessed it!) the golden ratio.

EXAMPLE 3 Using the Quadratic Formula

Solve $x^2-4x+2=0$.

SOLUTION

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ Quadratic formula

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(2)}}{2(1)}$$ $a=1, b=-4, c=2$

$$x=\frac{4\pm\sqrt{16-8}}{2}$$ Start to simplify.

$$x=\frac{4\pm2\sqrt{2}}{2}$$ $\sqrt{16-8}=\sqrt{8}=2\sqrt{2}$

$$x=\frac{2(2\pm\sqrt{2})}{2}$$ Factor out 2 in the numerator.

$$x=2\pm\sqrt{2}$$ Divide out common factor.

Factor, and then divide out the common factor.

The solution set is $\{2+\sqrt{2}, 2-\sqrt{2}\}$, abbreviated $\{2\pm\sqrt{2}\}$.

EXAMPLE 4 Using the Quadratic Formula

Solve $2x^2=x+4$.

SOLUTION

First write the equation in standard form as $2x^2-x-4=0$.

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4(2)(-4)}}{2(2)}$$ Quadratic formula with $a=2, b=-1, c=-4$

$$x=\frac{1\pm\sqrt{1+32}}{4}$$ Simplify the radicand.

$$x=\frac{1\pm\sqrt{33}}{4}$$ Add.

The solution set is $\left\{\frac{1\pm\sqrt{33}}{4}\right\}$.

Applications

EXAMPLE 5 Applying a Quadratic Equation

Two cars left an intersection at the same time, one heading due north, and the other due west. Some time later, they were exactly 100 miles apart. The car headed north had gone 20 miles farther than the car headed west. How far had each car traveled?

SOLUTION

Step 1 **Read** the problem carefully.

Step 2 **Assign a variable.**

Let x = the distance traveled by the car headed west.

Then $(x+20)$ = the distance traveled by the car headed north.

See **Figure 16**. The cars are 100 miles apart, so the hypotenuse of the right triangle equals 100.

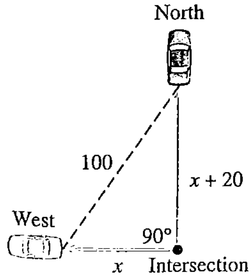


Figure 16

Évariste Galois (1811–1832), as a young Frenchman, agreed to fight a duel. He had been engaged for some time in mathematical research that centered on solving equations by using **group theory.** Now, anticipating the possibility of his death, he summarized the essentials of his discoveries in a letter to a friend. The next day Galois was killed. He was not yet 21 years old when he died.

Step 3 **Write an equation.**

$$c^2 = a^2 + b^2 \qquad \text{Pythagorean theorem}$$

$$100^2 = x^2 + (x + 20)^2 \qquad \text{Substitute.}$$

Step 4 **Solve.**

$$10{,}000 = x^2 + x^2 + 40x + 400 \qquad \text{Square the binomial.}$$

$$2x^2 + 40x - 9600 = 0 \qquad \text{Standard form}$$

$$x^2 + 20x - 4800 = 0 \qquad \text{Divide both sides by 2.}$$

Use the quadratic formula to find x.

$$x = \frac{-20 \pm \sqrt{20^2 - 4(1)(-4800)}}{2(1)} \qquad a = 1, b = 20, c = -4800$$

$$x = \frac{-20 \pm \sqrt{19{,}600}}{2} \qquad \text{Simplify under the radical.}$$

$$x = 60 \quad \text{or} \quad x = -80 \qquad \text{Use a calculator.}$$

Step 5 **State the answer.** Since distance cannot be negative, discard the negative solution. The required distances are 60 miles and $60 + 20 = 80$ miles.

Step 6 **Check.** Since $60^2 + 80^2 = 100^2$, the answer is correct.

Niels Henrik Abel (1802–1829) of Norway was identified in childhood as a mathematical genius but never received in his lifetime the professional recognition his work deserved. He proved that a general formula for solving **fifth-degree equations** does not exist. The quadratic formula (for equations of degree 2) is well known, and formulas do exist for solving third- and fourth-degree equations. Abel's accomplishment ended a search that had lasted for years.

EXAMPLE 6 Applying a Quadratic Equation

If a rock on Earth is projected upward from the top of a 144-foot building with an initial velocity of 112 feet per second, its position (in feet above the ground) is given by $s = -16t^2 + 112t + 144$, where t is time in seconds after it was projected. How long does it take for the rock to hit the ground?

SOLUTION

When the rock hits the ground, its distance above the ground is 0. Find t when s is 0.

$$0 = -16t^2 + 112t + 144 \qquad \text{Let } s = 0.$$

$$0 = t^2 - 7t - 9 \qquad \text{Divide both sides by } -16.$$

$$t = \frac{7 \pm \sqrt{49 + 36}}{2} \qquad \text{Quadratic formula}$$

$$t = \frac{7 \pm \sqrt{85}}{2} \qquad \text{Add.}$$

$$t \approx 8.1 \quad \text{or} \quad t \approx -1.1 \qquad \text{Use a calculator.}$$

This must be rejected.

Since the rock cannot hit the ground before it is projected, discard the negative solution. The rock will hit the ground about 8.1 seconds after it is projected.

7.7 EXERCISES

Fill in each blank with the correct response.

1. For the quadratic equation $5x^2 + 4x - 8 = 0$, the values of a, b, and c are, respectively, 5, 4, and −8.

2. To solve $3x^2 - 5x = -3$ by the quadratic formula, the first step is to add 3 to both sides of the equation.

3. Can the quadratic formula be used to solve the equation $2x^2 - 5 = 0$? Explain, and solve it if the answer is yes. Answers will vary. $\left\{\pm\frac{\sqrt{10}}{2}\right\}$

4. Can the quadratic formula be used to solve the equation $4x^2 + 3x = 0$? Explain, and solve it if the answer is yes. Answers will vary. $\left\{-\frac{3}{4}, 0\right\}$

Solve each equation by the zero-factor property.

5. $(x + 3)(x - 9) = 0$ $\{-3, 9\}$

6. $(x + 6)(x + 4) = 0$ $\{-6, -4\}$

7. $(2x - 7)(5x + 1) = 0$ $\left\{\frac{7}{2}, -\frac{1}{5}\right\}$

8. $(7x - 3)(6x + 4) = 0$ $\left\{\frac{3}{7}, -\frac{2}{3}\right\}$

9. $x^2 - x - 12 = 0$ $\{-3, 4\}$

10. $x^2 + 4x - 5 = 0$ $\{-5, 1\}$

11. $x^2 + 9x + 14 = 0$ $\{-7, -2\}$

12. $x^2 + 3x - 4 = 0$ $\{-4, 1\}$

13. $12x^2 + 4x = 1$ $\left\{-\frac{1}{2}, \frac{1}{6}\right\}$

14. $15x^2 + 7x = 2$ $\left\{-\frac{2}{3}, \frac{1}{5}\right\}$

15. $(x + 4)(x - 6) = -16$ $\{-2, 4\}$

16. $(x - 1)(3x + 2) = 4x$ $\left\{-\frac{1}{3}, 2\right\}$

Solve each equation by using the square root property. Give only real number solutions.

17. $x^2 = 64$ $\{\pm 8\}$

18. $x^2 = 16$ $\{\pm 4\}$

19. $x^2 = 24$ $\{\pm 2\sqrt{6}\}$

20. $x^2 = 48$ $\{\pm 4\sqrt{3}\}$

21. $x^2 = -5$ $\emptyset$

22. $x^2 = -10$ $\emptyset$

23. $(x - 4)^2 = 9$ $\{1, 7\}$

24. $(x + 3)^2 = 25$ $\{-8, 2\}$

25. $(4 - x)^2 = 3$ $\{4 \pm \sqrt{3}\}$

26. $(3 + x)^2 = 11$ $\{-3 \pm \sqrt{11}\}$

27. $(2x - 5)^2 = 13$ $\left\{\frac{5 \pm \sqrt{13}}{2}\right\}$

28. $(4x + 1)^2 = 19$ $\left\{\frac{-1 \pm \sqrt{19}}{4}\right\}$

Solve each equation by the quadratic formula. Give only real number solutions.

29. $4x^2 - 8x + 1 = 0$

30. $x^2 + 2x - 5 = 0$

31. $2x^2 = 2x + 1$

32. $9x^2 + 6x = 1$

33. $x^2 - 1 = x$

34. $2x^2 - 4x = 5$

35. $4x(x + 1) = 1$

36. $4x(x - 1) = 19$

37. $(x + 2)(x - 3) = 1$

38. $(x - 5)(x + 2) = 6$

39. $x^2 - 6x = -14$ $\emptyset$

40. $x^2 = 2x - 2$ $\emptyset$

41. Why can't the quadratic formula be used to solve the equation $2x^3 + 3x - 4 = 0$? The presence of $2x^3$ makes it a *cubic* equation (degree 3).

42. A student gave the quadratic formula incorrectly as follows: $x = -b \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$. What is wrong with this? The term $-b$ should be in the numerator, with denominator $2a$.

The expression $b^2 - 4ac$, the radicand in the quadratic formula, is called the ***discriminant*** *of the quadratic equation*

$$ax^2 + bx + c = 0, \quad a \neq 0.$$

By evaluating it we can determine, without actually solving the equation, the number and nature of the solutions of the equation. Suppose that a, b, and c are integers. Then the chart below shows how the discriminant can be used to analyze the solutions.

Discriminant	**Solutions**
Positive, and the square of an integer	Two different rational solutions
Positive, but not the square of an integer	Two different irrational solutions
Zero	One rational solution (a double solution)
Negative	No real solutions

In Exercises 43–48, evaluate the discriminant, and then determine whether the equation has **(a)** *two different rational solutions,* **(b)** *two different irrational solutions,* **(c)** *one rational solution (a double solution), or* **(d)** *no real solutions.*

43. $x^2 + 6x + 9 = 0$ 0; (c)

44. $4x^2 + 20x + 25 = 0$ 0; (c)

45. $6x^2 + 7x - 3 = 0$ 121; (a)

46. $2x^2 + x - 3 = 0$ 25; (a)

47. $9x^2 - 30x + 15 = 0$ 360; (b)

48. $2x^2 - x + 1 = 0$ −7; (d)

49. When using the quadratic formula, if $b^2 - 4ac$ is positive, then the equation has ___two___ (how many?) real solution(s).

50. If a, b, and c are integers in $ax^2 + bx + c = 0$ and $b^2 - 4ac = 17$, then the equation has ___two___ (how many?) irrational solution(s).

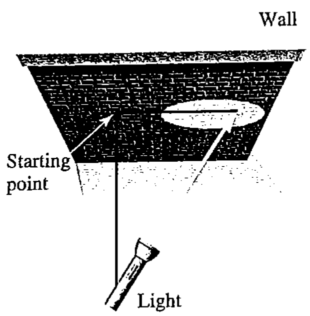
Solve each problem. Use a calculator as necessary, and round the answer to the nearest tenth.

51. ***Height of a Projectile*** A building is 400 feet high. Suppose that a ball is projected upward from the top, and its position s in feet above the ground is given by the equation $s = -16t^2 + 45t + 400$, where t is the number of seconds elapsed. How long will it take for the ball to reach a height of 200 feet above the ground? 5.2 seconds

52. ***Height of a Projectile*** A high-rise condominium building is 407 feet high. Suppose that a ball is projected upward from the top and its position s in feet above the ground is given by the equation $s = -16t^2 + 75t + 407$, where t is the number of seconds elapsed. How long will it take for the ball to reach a height of 450 feet above the ground? 0.7 second and 4.0 seconds

53. ***Height of a Projectile*** Refer to the equations in **Exercises 51 and 52.** Suppose that the first sentence in each problem did not give the height. How could you use the equation to determine the height? Find s when $t = 0$.

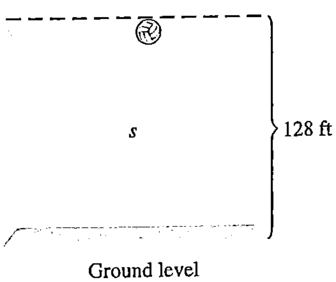
54. ***Position of a Flashlight Beam*** A flashlight beam moves horizontally back and forth along a wall with the distance of the light on the wall from a starting point at t minutes given by $s = 100t^2 - 300t$. How long will it take before the light returns to the starting point? 3 minutes



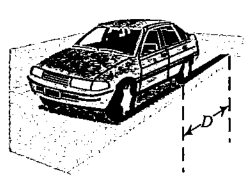
55. ***Height of a Projectile*** An object is projected directly upward from the ground. After t seconds its distance in feet above the ground is $s = 144t - 16t^2$.

(a) After how many seconds will the object be 128 feet above the ground? (*Hint*: Look for a common factor before solving the equation.) 1 second and 8 seconds

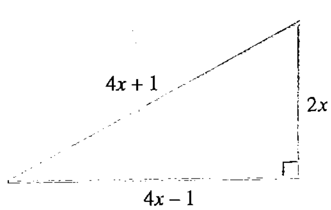
(b) When does the object strike the ground? 9 seconds after it is projected



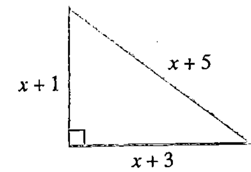
56. ***Distance of a Skid*** The formula $D = 100t - 13t^2$ gives the distance in feet a car going approximately 68 miles per hour will skid in t seconds. Find the time it would take for the car to skid 190 feet. (*Hint:* Your answer must be less than the time it takes the car to stop, which is 3.8 seconds.) 3.4 seconds



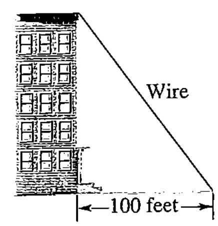
57. ***Side Lengths of a Triangle*** Find the lengths of the sides of the right triangle. 8, 15, 17



58. ***Side Lengths of a Triangle*** Find the lengths of the sides of the right triangle. 6, 8, 10

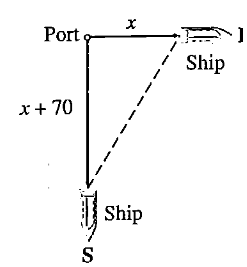


59. ***Length of a Wire*** Refer to **Exercise 51.** Suppose that a wire is attached to the top of the building and pulled tight. It is attached to the ground 100 feet from the base of the building, as shown in the figure. How long is the wire? 412.3 feet

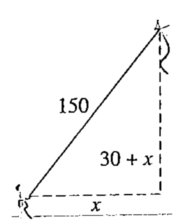


60. ***Length of a Wire*** Refer to **Exercise 52.** Suppose that a wire is attached to the top of the building and pulled tight. The length of the wire is twice the distance between the base of the building and the point on the ground where the wire is attached. How long is the wire? 470.0 feet

61. ***Distances Traveled by Ships*** Two ships leave port at the same time, one heading due south and the other heading due east. Several hours later, they are 170 miles apart. If the ship traveling south travels 70 miles farther than the other, how many miles does each travel? eastbound ship: 80 miles; southbound ship: 150 miles



62. ***Height of a Kite*** Paulette Starney is flying a kite that is 30 feet farther above her hand than its horizontal distance from her. The string from her hand to the kite is 150 feet long. How far is the kite above her hand? 120 feet

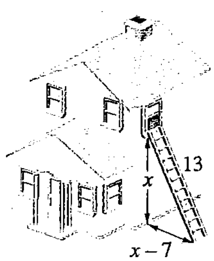


63. ***Size of a Toy Piece*** A toy manufacturer needs a piece of plastic in the shape of a right triangle with the longer leg 2 centimeters more than twice as long as the shorter leg, and the hypotenuse 1 centimeter more than the longer leg. How long should the three sides of the triangular piece be? 5 centimeters, 12 centimeters, 13 centimeters

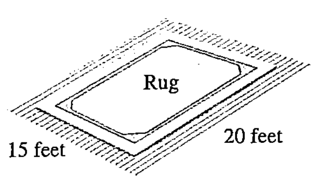
64. ***Size of a Developer's Property*** Hollis Sherman, a developer, owns a piece of land enclosed on three sides by streets, giving it the shape of a right triangle. The hypotenuse is 8 meters longer than the longer leg, and the shorter leg is 9 meters shorter than the hypotenuse. Find the lengths of the three sides of the property. 20 meters, 21 meters, 29 meters

65. ***Dimensions of Puzzle Pieces*** Two pieces of a large wooden puzzle fit together to form a rectangle with a length 1 centimeter less than twice the width. The diagonal, where the two pieces meet, is 2.5 centimeters in length. Find the length and width of the rectangle. length: 2 centimeters; width: 1.5 centimeters

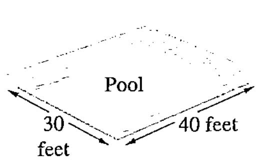
66. ***Leaning Ladder*** A 13-foot ladder is leaning against a house. The distance from the bottom of the ladder to the house is 7 feet less than the distance from the top of the ladder to the ground. How far is the bottom of the ladder from the house? 5 feet



67. ***Dimensions of a Strip of Flooring Around a Rug*** Kyle and Marin want to buy a rug for a room that is 15 feet by 20 feet. They want to leave an even strip of flooring uncovered around the edges of the room. How wide a strip will they have if they buy a rug with an area of 234 square feet? 1 foot



68. ***Dimensions of a Border Around a Pool*** A club swimming pool is 30 feet wide and 40 feet long. The club members want an exposed aggregate border in a strip of uniform width around the pool. They have enough material for 296 square feet. How wide can the strip be? 2 feet



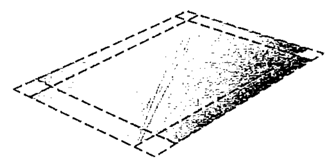
69. ***Dimensions of a Garden*** Arif's backyard is 20 meters by 30 meters. He wants to put a flower garden in the middle of the backyard, leaving a strip of grass of uniform width around the flower garden. Arif must have 184 square meters of grass. Under these conditions, what will the length and width of the garden be? length: 26 meters; width: 16 meters

70. ***Cardboard Box Dimensions*** If a square piece of cardboard has 3-inch squares cut from its corners and then has the flaps folded up to form an open-top box, the volume of the box is given by the formula

$$V = 3(x - 6)^2,$$

where x is the length of each side of the original piece of cardboard in inches. What original length would yield a box with a volume of 432 cubic inches? 18 inches

71. ***Dimensions of a Piece of Sheet Metal*** A rectangular piece of sheet metal has a length that is 4 inches less than twice the width. A square piece 2 inches on a side is cut from each corner. The sides are then turned up to form an uncovered box of volume 256 cubic inches. Find the length and width of the original piece of metal. length: 20 inches; width: 12 inches



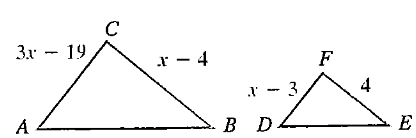
72. ***Interest Rate*** The formula

$$A = P(1 + r)^2$$

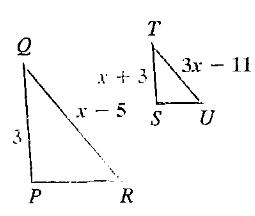
gives the amount A in dollars that P dollars will grow to in 2 years at interest rate r (where r is given as a decimal), using compound interest. What interest rate will cause \$2000 to grow to \$2142.25 in 2 years? 0.035, or 3.5%

Recall that the corresponding sides of similar triangles are proportional. (Refer to **Section 7.3 Exercises 53–56.**) *Use this fact to find the lengths of the indicated sides of each pair of similar triangles. Check all possible solutions in both triangles. Sides of a triangle cannot be negative.*

73. Side AC 5 or 14



74. Side RQ 4



75. For centuries mathematicians wrestled with finding a formula that could solve cubic (third-degree) equations. In sixteenth-century Italy, Niccolo Tartaglia had developed a method of solving a cubic equation of the form

$$x^3 + mx = n.$$

Girolamo Cardano begged to know Tartaglia's method, and after he was told, he was sworn to secrecy. Nonetheless, Cardano published Tartaglia's method in his 1545 work *Ars Magna* (although he did give Tartaglia credit).

The formula for finding one real solution of the equation $x^3 + mx = n$ is

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{-\left(\frac{n}{2}\right) + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

Solve $x^3 + 9x = 26$ using this formula. {2}

EXTENSION Complex Solutions of Quadratic Equations

Review of i and $\sqrt{-b}$, where $b > 0$ • The Discriminant • Solving a Quadratic Equation

Review of i and $\sqrt{-b}$, where $b > 0$ In the **Chapter 6 Extension,** we saw that the real number system is a subset of the complex number system. The imaginary unit i, where $i = \sqrt{-1}$, is defined so that $i^2 = -1$, and a number of the form $a + bi$, where a and b are real, is called a complex number.

A square root of a negative number can be written as the product of i and a real number. Here are some examples.

$$\sqrt{-4} = i \cdot \sqrt{4} = i \cdot 2 = 2i$$
$$\sqrt{-7} = i\sqrt{7}$$
$$\sqrt{-32} = i\sqrt{32} = i\sqrt{16 \cdot 2} = 4i\sqrt{2}$$

The Discriminant In the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

the expression under the radical symbol, $b^2 - 4ac$, is called the **discriminant.** In **Section 7.7,** we focused on quadratic equations with discriminants that were positive or zero. If the discriminant is negative, then the equation has two nonreal solutions.

Discriminant

The discriminant of $ax^2 + bx + c = 0$ is $b^2 - 4ac$. If a, b, and c are integers, then the number and type of solutions are determined as follows.

Discriminant	Number and Type of Solutions
Positive, and the square of an integer	Two rational solutions
Positive, but not the square of an integer	Two irrational solutions
Zero	One rational solution
Negative	Two nonreal solutions

EXAMPLE 1 Using the Discriminant

Show that $x^2 + x + 4 = 0$ has two nonreal complex solutions.

SOLUTION

$$\begin{aligned} b^2 - 4ac &= 1^2 - 4(1)(4) && \text{Here, } a = 1, b = 1, \text{ and } c = 4. \\ &= -15 && \textbf{Negative discriminant} \end{aligned}$$

Because $-15 < 0$, the equation has two nonreal solutions.

Solving a Quadratic Equation

EXAMPLE 2 Solving a Quadratic Equation with Nonreal Solutions

Solve $(9x + 3)(x - 1) = -8$.

SOLUTION

$$\begin{aligned} (9x + 3)(x - 1) &= -8 \\ 9x^2 - 6x - 3 &= -8 && \text{Multiply.} \\ 9x^2 - 6x + 5 &= 0 && \text{Add 8.} \end{aligned}$$

From the equation $9x^2 - 6x + 5 = 0$, we identify $a = 9$, $b = -6$, and $c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(5)}}{2(9)} \qquad \text{Substitute.}$$

$$x = \frac{6 \pm \sqrt{-144}}{18}$$

See the Chapter 6 Extension.

$$x = \frac{6 \pm 12i}{18} \qquad \sqrt{-144} = 12i$$

$$x = \frac{6(1 \pm 2i)}{6(3)} \qquad \text{Factor.}$$

$$x = \frac{1 \pm 2i}{3} \qquad \text{Lowest terms}$$

$$x = \frac{1}{3} \pm \frac{2}{3}i \qquad \text{Standard form } a + bi \text{ for a complex number}$$

The solution set is $\left\{\frac{1}{3} + \frac{2}{3}i, \frac{1}{3} - \frac{2}{3}i\right\}$.

EXTENSION EXERCISES

The following equations have nonreal solutions. Use the quadratic formula and the discussion of complex numbers in this ***Extension*** *and the* ***Extension in Chapter 6*** *to solve them.*

1. $x^2 + 12 = 0$ $\{-2i\sqrt{3}, 2i\sqrt{3}\}$

2. $x^2 + 18 = 0$ $\{-3i\sqrt{2}, 3i\sqrt{2}\}$

3. $9x(x - 2) = -13$ $\left\{1 + \frac{2}{3}i, 1 - \frac{2}{3}i\right\}$

4. $4x(x - 4) = -17$ $\left\{2 + \frac{1}{2}i, 2 - \frac{1}{2}i\right\}$

5. $x^2 - 6x + 14 = 0$ $\{3 + i\sqrt{5}, 3 - i\sqrt{5}\}$

6. $x^2 + 4x + 11 = 0$ $\{-2 + i\sqrt{7}, -2 - i\sqrt{7}\}$

7. $4x^2 - 4x = -7$ $\left\{\frac{1}{2} + \frac{\sqrt{6}}{2}i, \frac{1}{2} - \frac{\sqrt{6}}{2}i\right\}$

8. $9x^2 - 6x = -7$ $\left\{\frac{1}{3} + \frac{\sqrt{6}}{3}i, \frac{1}{3} - \frac{\sqrt{6}}{3}i\right\}$

9. $x(3x + 4) = -2$ $\left\{-\frac{2}{3} + \frac{\sqrt{2}}{3}i, -\frac{2}{3} - \frac{\sqrt{2}}{3}i\right\}$

10. $x(2x + 3) = -2$ $\left\{-\frac{3}{4} + \frac{\sqrt{7}}{4}i, -\frac{3}{4} - \frac{\sqrt{7}}{4}i\right\}$